Supergravity in superspace
supergeometry, differential forms and algebraic structure

Greitz, Jesper

Awarding institution:
King's College London

Title: Supergravity in superspace: supergeometry, differential forms and algebraic structure

Author: Jesper Greitz

Supergravity in superspace: supergeometry, differential forms and algebraic structure

Jesper Greitz

Thesis submitted for the degree
Doctor of Philosophy in Mathematics

Abstract

The following thesis will be concerned with various aspects of supergravity theories in a superspace setting, focusing mainly on maximal and half-maximal theories in three dimensions and maximal theories in ten-dimensions. For the three-dimensional theories it is convenient to start from an off-shell superconformal geometry valid for any number of supersymmetries. We first apply this formalism to show that it is consistent to couple ABJM and BLG theory to conformal supergravity, in doing so we find that $N = 8$ superconformal matter can also be charged under the gauge group $SO(N)$.

By imposing further constraints on the off-shell superconformal geometry, we obtain half-maximal and maximal Poincaré supergravity. We solve for the geometry at dimension one in the half-maximal case with sigma models of the form $(SO(8) \times SO(n))\backslash SO(8, n)$, and for the complete geometry in the maximal theory, where the scalar fields live in the coset $SO(16)\backslash E8$. Using the Ricci identity, we also derive the equations of motion for the scalar and fermion fields in the latter theory.

Using supersymmetry and duality we derive the form spectrum of the above Poincaré supergravity theories and of type IIA and IIB supergravity in ten dimensions. Particular we show that the consistent Bianchi identities, which are not guaranteed to be satisfied from cohomology, determine a Lie super co-algebra. We derive the Cartan matrices of the dual algebras which are Borcherds algebras. The Cartan matrices can be used to generate the entire form field spectrum.

We study gaugings of half-maximal and maximal Poincaré supergravity in three dimensions by introducing a non-abelian gauged subgroup of the duality group and making use of the gauged Maurer-Cartan form. The differential forms can also be studied in the gauged theory by deforming the Bianchi identities. The closure of the full system of forms requires the presence of

$D+2$-form field strengths in the supergravity limit.

In superspace, the Borcherds algebras predict an infinite number of form fields of degree larger than that of space-time. Indeed all those of degree larger than $D+2$ are zero in supergravity, although this might change in string theory. We provide some evidence that a six-form, in half-maximal supergravity in three dimensions can become non-zero in the presence of α'-corrections.

Contents

List of Tables

Chapter 1

Introduction

Perhaps the most successful idea in physics is that all observers should agree on the laws of nature. It is said that Galileo Galilei reached this conclusion by noting that it is impossible for someone travelling on a completely calm ocean, on a boat which leaves no trace behind, to determine whether he is moving or not, yet, he observes the same physical laws as on land. The laws of nature must reflect this, and possess symmetries which ensure they are independent of the coordinate system being used. These symmetries are called spacetime symmetries, and any theory must be invariant under their corresponding group action. This principle is used in both Newton's laws and in relativistic physics but their spacetime-symmetry group is not the same. Newton's laws of motion are invariant under transformations of the Galilean group, which realises the relative nature of space, but asserts the everyday conception of time as separated from it, as something ticking constantly in the background. When it became clear that the speed of light does not depend on whether you are moving away from, or towards the light source, Albert Einstein understood that not only space but also time is relative. and his theory of special relativity is instead invariant under transformations of the Poincaré group.

Galilei's idea leads to far-reaching conclusions of how we think of the most fundamental objects in nature. It is of course unavoidable that the particles are elements of the theories describing them. As such, they need to have well-behaved transformations under the space-time symmetry group, so that an invariant theory of their dynamics and interactions can be constructed. In other words, they need to be objects which transform under representations of the space-time symmetry group, which is also the modern understanding of point particles.

Nature has another type of symmetries, internal symmetries. They are symmetries of the particles themselves and provide constraints on the interactions. Their origin is not so easily understood, nevertheless the standard model, a theory based on local $SU(3) \times SU(2) \times U(1)$ internal gauge symmetry and global Poincaré symmetry, has been successful in describing the electro-weak and strong interactions. It has predicted the existence of three particles, the W and Z boson and the Higgs particle. The W and Z bosons, the force carriers of the weak nuclear interactions, were detected experimentally some time ago while recent results obtained at the LHC suggest the existence of the Higgs particle with a mass of around 125 GeV. However, there are many questions of theoretical interest that the standard model leaves out such as why the specific gauge groups appear and why the many free parameters take the values they do. Whether it is possible to find a more fundamental theory answering these questions is not known; but even if we believe there need not exist such a theory, the standard model falls short in other respects. There are for example hints from cosmology that most of the matter in the universe is not found in the standard model, although this matter, called dark matter is not directly observable, its existence is assumed in order to explain the motion of stars in galaxies. More importantly, perhaps, is the fact that the standard model leaves out gravity. The standard theory of gravity, Einstein's general relativity, is treated as a classical theory and it is not known whether it can be consistently quantised. One possible attempt to reconcile this is to look for more, or larger symmetries in nature. This could hopefully lead to a more general version of the standard model or general relativity which either combine them, or in some other way cure the divergences found in quantum gravity. The hope to find such symmetries partly comes from the relative strengths of the different forces in the standard model. The coupling constants, which determine the strengths, depend on the energy of the particles undergoing interactions. At 10^{16} GeV the forces of the standard model are almost equally strong, which might suggests that the gauge groups combine into one. Apart from larger gauge groups there is another symmetry which has attracted a lot of interest in theoretical physics.

The Coleman-Mandula no-go theorem [1] states that the only Lie-algebraic symmetries of the S-matrix must be a direct sum of space-time and compact internal Lie algebras. This theorem relies on a number of assumptions and by relaxing one of them to allow for graded commutators this conclusion can be avoided. The algebra which extends the Poincaré algebra with anti-commuting generators is called the supersymmetry algebra [2], [3]. It mixes internal and space-time symmetries and this was taken as one of the motifications for supersymmetry when it first appeared. However, so far there has not been much success in creating realistic models with internal symmetries entering this way. Representations of the supersymmetry algebra contain particles of different spins in such a way that the bosonic and fermionic degrees of freedom always match up. This leads supersymmetric theories to have better ultra-violet properties than their non-supersymmetric versions due to cancellations between fermionic and bosonic contributions. As an example we can give the maximally supersymmetric ($N = 4$) version of Yang-Mills theory in four dimensions [4] which is ultra-violet finite [5], [6], [7].

Supersymmetry can also be used to generalise Einstein's theory of gravity. The commutator of two supersymmetry transformations gives a translation, so that by promoting supersymmetry to a local symmetry we naturally get a supersymmetric version of general relativity, called supergravity [8]. For pure supergravity theories, the number of supersymmetry generators determines the particle spectrum. Common to all of them is the presence of the graviton and its spin three-halves superpartner the gravitino, the gauge particle for supersymmetry. Extended supergravity theories, which use more than one set of supersymmetry generators, contain additional particles and include one gravitino for each supersymmetry. In the early days there were hopes that an extended version of supergravity could provide a unified theory for all forces, a not unreasonable proposal since such models have particle spectra that combine gravitons with gauge bosons, fermions and scalar particles. However, so far there has not been any success incorporating the gauge groups of the standard model into a supergravity theory. It is also widely believed that supergravity is not UV finite, although recent developments in computational techniques have allowed explicit calculations to be carried out at higher loop-orders than had previously been thought possible [9]. These computations have shown that the onset of ultra-violet divergences occurs at a higher loop order than had originally been supposed. It is now thought, on symmetry grounds, that the first divergences are likely to occur at 7 or perhaps 8 loops [10], [11], [12], although there is a minority view that $N = 8$ supergravity might be finite to all orders [13], [14].

An earlier attempt to unify gravity and electromagnetism made in the twenties was made by constructing a theory of gravity in a space-time which instead of three, had four spatial directions [15], [16]. If one of these is taken to be very small, the theory in five dimensions (four space and one time) can be reinterpreted as a unified description of gravity and electromagnetism in four dimensions. Similarly, we can formulate supergravity in any number of dimensions, but if we insist on not having particles of spin larger than two, the number of spatial directions is restricted to be at most ten. In eleven dimensions there is a unique theory of supergravity [17], and with decreasing dimensions there is an increasing number of possibilities, many of which can be reached by dimensionally reducing the theory in eleven dimensions. This was also how the maximally supergravity theory in four dimensions was first constructed [18]. It came as a surprise to many people that the theory has a hidden global $E7$ symmetry. It is now known that all maximal supergravity theories from three up to nine dimensions have a global symmetry in the E series. The reason for their appearance is not well understood but a possible explanation might come from string theory.

String theory differs in quite a fundamental way from point-particle theories. The basic idea of string theory is that the fundamental elements of nature are vibrating strings. To consistently write down a theory for strings one needs to introduce supersymmetry and the dimension of space-time must be ten. After the first string revolution in the mid eighties [19] it became clear that there are five consistent superstring theories. After another decade of research it was understood that the five theories were related by various dualities [20]. This led to a famous conjecture that they are all different corners of a unique theory in eleven dimensions, M-theory. The way supergravity and superstring theory fit together can be seen by considering the low energy limit of the latter. For distances which are large with respect to the string length we can think of strings as point particles. In this first-order approximation, the five perturbative superstring theories are supergravity theories. It is also conjectured that the low energy limit of M-theory is the unique eleven-dimensional supergravity theory. In contrast to supergravity, superstring theory can contain the gauge groups of the standard model; in fact it allows for a huge number of different universes and one of the most challenging problems is to find our own. It is also thought that superstring theory provides a finite quantum theory of gravity. Having such a promising but difficult theory the study of its first order approximation is very much worthwhile. Supergravity, and how it changes when corrections are added according to string theory is therefore still an active area of research. The hidden symmetries found in supergravity theories are related to the duality symmetries in superstring theory, which is why they are most often referred to as duality symmetries.

This thesis will be concerned with supergravity theories in superspace. Superspace is the most natural arena for supersymmetric theories, in that the supersymmetry algebra acts on it in a natural fashion. A point in superspace is parameterised by both even and odd coordinates and the supersymmetry transformations act as translations on the odd directions of superspace. We can think of flat superspace in a similar way to how we think of Minkowski space as the coset of the Poincaré- group divided by the Lorentz group. Taking this point of view superspace is the super Poincaré group divided by the Lorentz group. Even for rigid supersymmetry superspace is not flat because two supersymmetry transformations give rise to a spacetime translation. There are both advantages and drawbacks to using superspace. Advantages include the fact that it is possible to give a complete geometrical interpretation of supergravity and that it is always possible to work in a completely covariant formulation. The main disadvantage is that it becomes necessary to introduce many more fields than the theory requires, and to remedy this one needs to get rid of the extra fields by putting constraints on the superfields. Historically, however, supersymmetric theories have often been constructed first in space-time and then re-cast into a superspace formulation.

Using superspace we will discuss ungauged and gauged supergravity theories in three-dimensional spacetime. Gravity theories in three dimensions have been studied for a long time [21] and quite often to serve as toy models, but they also have some interesting features by themselves. In three dimensions the Riemann tensor vanishes on-shell in Einstein's theory and as consequence there are no propagating degrees of freedom [22], although perhaps surprisingly, in the presence of a negative cosmological constant there exists black hole solutions [23]. Another interesting feature in three dimensions is that one can formulate the action for a gravitational theory using a Chern-Simons term [24], [25]. Furthermore by combining the Chern-Simons action with the Einstein-Hilbert action one gets a theory known as topological massive gravity [26]. The theory, which contains a propagating massive spin-two particle and black holes, was recently shown to avoid negative energy modes for certain values of its free parameters despite it having a higher-derivative action [27]. There also exists another ghost-free but unitary higher-derivative theory named new massive supergravity [28]. Although the price of this theory being unitary is that it is non-renormalisable, something that might be overcome by considering its supersymmetric extensions [29]. As previously mentioned, our main focus will be on supergravity theories in three dimensions [31], [30], [32]. From a superspace perspective three dimensions is interesting also because it is straightforward to construct an off-shell superconformal geometry valid for any number of supersymmetry generators. This makes the geometry useful as a starting point to discuss various supersymmetric theories.

In chapter 3 it is used to study superconformal matter theories related to $M2$-branes [33], [34], [35]. By coupling $N = 8$ superconformal matter to a conformal supergravity background we show there is more freedom in the choice of gauge groups than in the absence of gravity, although the physical interpretation of such theories is not completely clear. The $N = 6$ models are also briefly discussed.

Starting from the off-shell superconformal geometry we also show that by coupling it to sigma models we can derive the geometry of half-maximal and maximal on-shell Poincaré supergravity theories. In particular we derive the complete geometry for the maximal case, which is interesting since it carries a global $E8$ symmetry, the largest of all exceptional Lie groups. In two and one dimensions the duality algebras of maximal supergravity have been conjectured to be the infinite Lie algebras $E9$ and $E10$ [36].

We then go on to derive the form fields for the three-dimensional theories discussed above and also those of type IIA and IIB supergravity. We derive these in a very simple manner using only supersymmetry and duality. The derivation is greatly facilitated by the use of superspace cohomology and the completely covariant formalism available in superspace. The latter is possible since odd basis forms are commutative meaning that the degree of a form can be larger than the dimension of spacetime. One can therefore construct a covariant $D+1$-form field strength for the D-form potentials. Using superspace cohomology the "identities for identities" tells us that if a few components for some of the Bianchi identities for the lower degree form fields are satisfied, then so are the consistent Bianchi identities for all the higher rank forms.

There has been a considerable interest in the Kac-Moody algebras [37], [38] and Borcherds algebras [39] which have been shown to be related to the form fields in maximal supergravity theories. The study of the algebraic properties of the form fields started in the nineties. It became clear that the algebras governing the structure of the form fields are Borcherds algebras and that all maximal supergravity theories have a Borcherds algebra associated with them. Borcherds algebras has also appeared in the hierarchy of forms in gauged maximal supergravity and in the ghost structure of the algebra of generalised diffeomorphisms [40] in the doubled field theory formalism [41]. A separate observation is that the infinite groups $E10$ [42] and $E11$ [43] predict the form fields of all maximal supergravity theories under relevant decompositions and this led to speculations whether they are symmetries of a more fundamental underlying theory, such as M-theory. However the decompositions of $E10$ and $E11$ contain an infinite number of modules which do not correspond to form fields and the interpretation of these are not clearly understood.

Borcherds algebras are particularly interesting to consider in superspace. The reason being that they predict form fields of arbitrarily high rank which fits well with the superspace formulation. We investigate whether the forms in type IIA and IIB supergravity with degree larger than spacetime are those predicted by Borcherds algebras. Using half-maximal supergravity we go on to ask why Borcherds algebras appear in supergravity theories and we show that the form fields of this theory are also encoded by Borcherds algebras. The appearance of the Borcherds algebras in supergravity theories can be easily understood in superspace. By combining our results with recent results on the relation between Borcherds and Kac-Moody algebras [44] we can also understand why the Kac-Moody algebras $E10$ and $E11$ correctly predict the form fields in supergravity theories.

Most of the form fields of degree larger than spacetime vanish identically in supergravity, but in principle this can change when string corrections are added. In half-maximal supergravity corrections start already at order α' and the duality group is more easy to handle than $E8$ which is why we have chosen to work with this theory. This will be the topic of the final chapter where we investigate whether the form fields predicted by the Borcherds algebras can be non-zero in string theory.

Chapter 2

Conventions, algebras and representations

In the following chapter we discuss properties of particles and supermultiplets in three-dimensional Minkowski space. We begin with a brief presentation of the algebras underlying the theories being constructed later on. The discussion will exclusively be in $2+1$ dimensions as most of the thesis belongs here; whenever we will use another dimension, or a Lie algebra, our conventions can be found in appendix A or nearby. Before turning to the algebras, we give our space-time conventions.

2.1 Space-time conventions

We take the Minkowski metric to be mostly positive $\eta_{ab} = \mathrm{diag}(-1,1,1)$. The epsilon tensor is defined so that $\varepsilon_{012} = +1$, one can use the epsilon tensor to construct the dual to a one-form, $v_{ab} := \varepsilon_{abc}v^c$, or inversely $v_a = -\frac{1}{2}\varepsilon_{abc}v^{bc}$.

The gamma-matrices with their indices in standard position are $(\gamma^a)_\alpha{}^\beta$ (α=1,2). From these we can form $\gamma_{(a}\gamma_{b)} = \eta_{ab}$, $\gamma_{[a}\gamma_{b]} = \gamma_{ab}$, $\gamma_{[a}\gamma_b\gamma_{c]} = \gamma_{abc}$, F $\gamma_{ab} =$

$\varepsilon_{abc}\gamma^c$ and $\gamma_{abc} = \varepsilon_{abc}$. When multiplying the gamma matrices they are assumed to have their indices in their standard position. Spinor indices are lowered or raised with the spin "metrics" $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ which we take to have the same numerical entries, i.e. $\varepsilon_{12} = \varepsilon^{12} = +1$. The summation convention is NE-SW, i.e. $v^\alpha = \varepsilon^{\alpha\beta} v_\beta$ and $v_\alpha = v^\beta \varepsilon_{\beta\alpha}$. The matrices γ_a (and γ_{ab}) with both spinor indices down (or up) are symmetric.

Bi-spinors can be expanded in terms of the gamma matrices. The completeness relation

$$M_{\alpha\beta} = {\gamma^a}_{\alpha\beta} M_a + \varepsilon_{\alpha\beta} M_0 \tag{2.1.1}$$

allows us to write vectors as a symmetric bi-spinor or vice-versa

$$v_{\alpha\beta} = {\gamma^a}_{\alpha\beta} v_a \Leftrightarrow v_a = -\frac{1}{2} {\gamma_a}^{\alpha\beta} v_{\alpha\beta}. \tag{2.1.2}$$

For any two spinors ψ, χ and any gamma-matrix Γ we define the tensorial bilinear to be

$$\psi\Gamma\chi := \psi^\alpha {\Gamma_\alpha}^\beta \chi_\beta \ . \tag{2.1.3}$$

The gamma-matrices satisfies

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{2.1.4}$$

We can choose a real basis for the gamma matrices in three dimensions such that $\gamma^\mu = \{i\sigma^2, \sigma^1, \sigma^3\}$, where σ are the Pauli matrices

$${\gamma^0}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ {\gamma^1}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ {\gamma^2}_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

lowering the raised index using the epsilon tensor we get,

$$\gamma^0{}_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \ \gamma^1{}_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ \gamma^2{}_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0. \end{pmatrix},$$

There are plenty of relations that can be derived from (2.1.4), particularly the identity

$$\gamma^a{}_{\alpha\beta}\gamma_{a\gamma\delta} = 2\varepsilon_{\alpha(\gamma}\varepsilon_{\delta)\beta} \tag{2.1.5}$$

is useful. We also summarise some other basic relations

$$\begin{aligned} \gamma^a\gamma^b &= \gamma^{ab} + \eta^{ab} \\ \gamma^a &= -\frac{1}{2}\varepsilon^{abc}\gamma_{bc} \\ \gamma^{ab} &= \varepsilon^{abc}\gamma_c \ . \end{aligned} \tag{2.1.6}$$

2.2 Algebras

2.2.1 Poincaré algebra

Any physical theory that obeys the principles of special relativity should be invariant under transformations of the Poincaré algebra. In three-dimensional space-time the Poincaré algebra is

$$\begin{aligned}
[P_a, P_b] &= 0 \\
[J_a, P_b] &= \varepsilon_{abc} P^c \\
[J_a, J_b] &= \varepsilon_{abc} J^c .
\end{aligned} \tag{2.2.1}$$

The Poincaré algebra realises coordinate transformations such that the proper time interval $d\tau = x^a x^b \eta_{ab}$ is left invariant. J^0 is the generator of Lorentz rotations in the spatial plane, J^1 and J^2 generate Lorentz boosts in the two spatial directions, while P_a generate translations.

2.2.2 Conformal algebra

The conformal algebra loosens the defining constraint of the Poincaré algebra and allows for transformations leaving the proper time interval invariant up to a scale factor. The conformal Poincaré algebra in three-dimensional Minkowski space is $SO(2,3)$ whose Lie algebra is

$$[M_{rs}, M_{tu}] = -4\eta_{[r[t} M_{s]u]} \ . \tag{2.2.2}$$

M is antisymmetric, $r = (0, 1..., 4)$ and $\eta_{rs} = \mathrm{Diag}(-1, +1, +1, +1, -1)$. By re-labeling the generators in the following way, $M_{ab} = \varepsilon_{abc} J^c$, $M_{a3} = \frac{1}{2}(P_a - K_a)$, $M_{a4} = \frac{1}{2}(P_a + K_a)$ and $M_{34} = D$, the algebra reduces to (2.2.1) together with the following commutation relations

$$\begin{aligned}
[D, P_a] &= -P_a & [P_a, K_b] &= -2(\eta_{ab} D - \varepsilon_{abc} J^c) \\
[D, J_a] &= 0 & [J_a, K_b] &= \varepsilon_{abc} K^c \\
[D, K_a] &= K_a & [K_a, K_b] &= 0 \ .
\end{aligned} \tag{2.2.3}$$

D is the generator of dilations and K_a are the generators of special conformal transformations. Dilations are scaling transformations and a special conformal transformation is a combination of a translation and an inversion. A theory which is invariant under transformations of the conformal group is not sensitive to distances, only to angels.

2.2.3 Super-Poincaré algebra

The commutation relations of the super-Poincaré algebra in three dimensions are those of the Poincaré algebra (2.2.1) together with the following

$$\begin{aligned} \{Q_{\alpha i}, Q_{\beta j}\} &= -\delta_{ij}\gamma_{\alpha\beta}P_a \\ \left[P_a, Q^i_\alpha\right] &= 0 \\ [J_a, Q_{\alpha i}] &= -\frac{1}{2}\gamma_{a\alpha}{}^{\beta}Q_{\beta i}\ . \end{aligned} \tag{2.2.4}$$

$Q_{\alpha i}$ are supersymmetry generators, and for N-extended supersymmetry, i takes the values $1, 2..., N$. In three dimensions Q is a real two-component spinor, making the total number of supersymmetry generators add up to $2N$.

2.2.4 Superconformal algebra

The superconformal algebra in three dimensions is $OSp(2|N)$. Its bosonic part consists of $Sp(2)$ which is isomorphic to $O(2,3)$, the conformal group in three-dimensional Minkowski space, and $O(N)$ which is the R-symmetry group related to rotations of the supersymmetry generators. Apart from the Q-supersymmetry generators the superconformal algebra also have S-supersymmetry generators. The commutation relations are those of the conformal algebra (2.2.3), the super-Poincaré algebra (2.2.4) together with the following relations

$$
\begin{array}{rclrcl}
[P_a, S_{\alpha i}] &=& -\gamma_{a\alpha}{}^{\beta} Q_{\beta i} & [Q_{\alpha i}, S_{\beta j}] &=& \delta_{ij}\varepsilon_{\alpha\beta} D - \delta_{ij}\gamma^{a}{}_{\alpha\beta} J_a - \varepsilon_{\alpha\beta} N_{ij} \\
[J_a, S_{\alpha i}] &=& -\frac{1}{2}\gamma_{a\alpha}{}^{\beta} S_{\beta i} & [S_{\alpha i}, S_{\beta j}] &=& -\delta_{ij}\gamma^{a}{}_{\alpha\beta} K_a \\
[D, S_{\alpha i}] &=& \frac{1}{2} S_{\alpha i} & [K_a, Q_{\alpha i}] &=& -\gamma_{a\alpha}{}^{\beta} Q_{\beta i}, \\
[K_a, S_{\alpha i}] &=& 0 & [D, Q_{\alpha i}] &=& -\frac{1}{2} Q_{\alpha i}
\end{array}
\tag{2.2.5}
$$

where N_{ij} are the generators of $O(N)$. The S-supersymmetry generators share the properties of the Q-supersymmetry generators except that they instead square to special conformal transformations.

2.3 Representations

Having briefly introduced the relevant algebras, we will now discuss the properties of particles in $2+1$-dimensions and the particle content of supersymmetric theories. In a quantised theory we think of point particles as unitary irreducible representations (UIR) of the Poincaré group. A systematical study of the UIRs in $2+1$-dimensions was carried out in [46]. The author studied all representations of the Poincaré group, however, we will not consider the UIRs of the Poincaré group corresponding to particles of negative energy or continuous spin. The occurrence of negative energy modes, also known as tachyons, is seen as an instability of a physical theory. Continuous spin representations are generally not thought to be of relevance and has not been observed in nature, although they have been considered in the literature, for a discussion see for example [47].

2.3.1 Massive representations of the Poincaré algebra

The UIRs corresponding to massive particles can be analysed in the particles rest-frame where $P^a = (m, 0, 0)$. In matrix notation this reads

$$P^a\gamma_a = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}. \tag{2.3.1}$$

The Lorentz transformations leaving the particle in this frame are those for which

$$L \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} L^T = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}. \tag{2.3.2}$$

These transformations form a subset of all Lorentz transformations satisfying $L^T = L^{-1}$ which is the defining relation of $O(2)$. There are two eigenvectors associated to transformations of the UIR of $O(2)$. It is therefore possible to classify massive particles using the notion of positive and negative helicity. In this way, massive particles in three dimensions resemble massless particles in $3+1$ dimensions.

2.3.2 Massless representations of the Poincaré algebra

In the case of massless particles P squares to 0. Using our freedom to choose an appropriate coordinate system we go to the frame where $P^a = (w, -w, 0)$ or equivalently where

$$P^a\gamma_a = \begin{pmatrix} -2w & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.3.3}$$

The Lorentz transformations leaving this frame invariant are those for which

$$L = \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} . \tag{2.3.4}$$

These form a direct product of $\mathbb{Z} \otimes \mathbb{R}$. Their UIRs are given by their trivial representations and $\pm I$ for $\mathbb{Z}$ and e^{at} for $\mathbb{R}$, where a is a pure imaginary number and t a real parameter specifying the group element. It is not possible to classify states transforming under these representations in terms of helicities. The trivial representation of $\mathbb{R}$ is the one relevant for us. The trivial representation of $\mathbb{Z}$ correspond to massless bosonic particles, while the $\pm\mathbb{I}$ representation gives a massless spinor containing only one physical state.

2.4 Supermultiplets

A UIR of the supersymmetry group is a collection of UIRs of the Poincaré group known as a supermultiplet. All particles of a supermultiplet share the same mass since P^2 is a Casimir operator of the super-Poincaré algebra. However, the square of the Pauli Lubanski n-form, which in three dimensions is the scalar $P^a J_a$, is no longer a Casimir operator. This means that a supermultiplet contains particles of different spin, however not in an arbitrary way, but always such that there are as many bosonic as fermionic degrees of freedom.

2.4.1 Massive supermultiplets

The massive supermultiplets in three dimensions satisfies[1] $P^2 = M^2$. In the particles rest-frame, where $P_\mu = (m, 0, 0)$ the supersymmetry algebra is

$$\{Q^i_\alpha, Q^j_\beta\} = 2m\delta_{ij}\gamma^0_{\alpha\beta}. \tag{2.4.1}$$

We can recast the above algebra into the following form

[1]Here we follow closely the discussion given in [48]

$$\{a^i, (a^j)^\dagger\} = \delta^{ij} \tag{2.4.2}$$
$$\{a^i, a^j\} = \{(a^i)^\dagger, (a^j)^\dagger\} = 0, \tag{2.4.3}$$

where $a^i = Q_1 - iQ_2$. Using the Lorentz generator J^0 (2.2.1) to measure the particles helicity we find that $a^i((a^i)^\dagger)$ lowers (raises) the spin by $\frac{1}{2}$. States can be formed by acting with these operators on the vacuum state $|\Omega>$ carrying spin λ

$$a_+^{i_1}...a_+^{i_n}|\Omega> . \tag{2.4.4}$$

Because of the anticommuting property of the creation operators, we count that there are $\begin{pmatrix} N \\ n \end{pmatrix}$ states of spin $\lambda + \frac{1}{2}n$. We note that the massive supermultiplets in $2+1$ dimensions are the same as the massless supermultiplets in $3+1$ dimensions.

2.4.2 Massless supermultiplets

Massless supermultiplets in three dimensions have been studied in [49] and we briefly reproduce their discussion. For massless representations $P^2 = 0$. To determine the supermultiplets it is useful to go to a frame where $P_\mu = (-E, E, 0)$. In this frame the supersymmetry algebra reads

$$\{Q_\alpha^i, Q_\beta^j\} = 2E\delta_{ij}(\gamma^0_{\alpha\beta} + \gamma^1_{\alpha\beta}) . \tag{2.4.5}$$

This can be spit into

$$\{a_1^i, a_1^j\} = \delta_{ij} \tag{2.4.6}$$

$$\{a_1^i, a_2^j\} = \{a_2^i, a_2^j\} = 0, \tag{2.4.7}$$

by denoting $a_\alpha^i = \frac{1}{\sqrt{2E}} Q_\alpha^i$. (2.4.6) defines a Clifford algebra with no time-like directions. Representations of Clifford algebras in various dimensions with different space-time signatures have been studied extensively and a good reference is [50]. The representations of the Clifford algebra defined in (2.4.6) is given in table 1.

N	1	2	3	4	5	6	7	8	n+8
d_N	1	2	4	4	8	8	8	8	n+8

Table 2.1: The table presents the irreducible massless supermultiplets in three-dimensional Minkowsi space. N denotes the number of supersymmetry generators and d_N the number of bosonic states.

As there is no notion of helicity for massless particles in three dimensions, both the scalar and spinor fields carry one physical degree of freedom.

Chapter 3

Superconformal gravity

The theory of $N = 1$ conformal supergravity (CSG) in three dimensions was first constructed in [51], and the extension to larger N came soon afterwards [52]. The construction of these theories followed closely that of $N = 1$ CSG in four dimensions [53] where the generalisation to CSGs for $N \leq 4$ was done in [54]. An interesting feature of the $N = 4$ CSG coupled to maximal super Yang-Mills theory with gauge groups $SU(2) \times U(1)$ is that it is a finite and anomaly free theory [55], even though it is unclear whether it can reduce to a phenomenologically acceptable theory at low energies [56].

The CSG theories were constructed by gauging the entire superconformal group and then constraining the curvatures such that local Poincaré transformations generate general coordinate transformations. In the superspace approach to CSG [57], only the Lorentz- and R-symmetry group is explicitly gauged. The P, Q and K transformations are absorbed into the general coordinate transformations, while the S-supersymmetry transformations and dilations become hidden symmetries of the geometry. The hidden symmetries arise from variations of the supervielbeins and the superconnection which leaves the constraint on the dimensions-zero torsion invariant. This is the only constraint that is put in by hand, the other components of the torsion and curvature follow solely from the Bianchi identities. The hidden symmetries are parameterised by an unconstrained scalar superfield which we can think of as the super-Weyl parameter. When we have solved for the geometry, having put in the constraints, many of the fields are auxiliary because of the hidden symmetry.

More generally, this is how supergravity theories are constructed in superspace. By choosing some suitable constraints on the torsion and curvature tensors, one wishes to obtain either an on-shell or off-shell description of the geometry. The wrong set of constraints lead to inconsistencies or unwanted equations of motions. It is quite often not clear what the interpretation of a particular set of constraints are [58], [59], and the rôle of the Bianchi identities is to determine this. They also tell you whether two chosen constraints are compatible, since far from all components are independent. A famous example of this is Dragon's second theorem [60] which states that for $D \geq 4$ the curvature is determined by the components of the torsion. The use of superspace cohomology also simplifies the analysis of the Bianchi identities since they reduce the number of Bianchi identities one needs to show are satisfied.

An example that illuminates the difficulty of knowing the consequences of a certain constraint is the following. Choosing the dimensions-zero torsion to coincide with that of flat superspace in eleven dimensions [61] leads to an on-shell description of the unique supergravity theory in eleven dimensions [17], while in three dimensions the same constraint leads to the superconformal geometry valid for any number of supersymmetry generators discussed in this chapter. This should not be too surprising since the constraint in eleven dimensions is much stronger. If we were to dimensionally reduce the constraint to three dimensions, we would get many more constraints than the previously mentioned.

We will begin with an introduction to superconformal gravity in superspace. The superspace geometry is valid for any N although we also give an explicit discussion of the $N = 6$ and $N = 8$ geometry. We then consider the $N = 6$ and $N = 8$ superconformal matter multiplets of ABJM [33] and BLG [34], [35] theory. We use the Lie-algebra formalism [62] to give a superspace derivation that reproduces the well-known result that these multiplets can only be charged under $U(N) \times U(N)$ and $SU(2) \times SU(2)$ respectively. Coupling these theories to a superconformal background give no further possibilities for the $N = 6$ multiplet, for $N = 8$ however, we find that there is also the possibility of having $SO(N)$ as a gauge group. The interpretation of these theories in terms of M2 branes is not entirely clear. In [63] it was argued that by coupling BLG theory to $N = 8$ CSG one allowed for a mixture of Dirichlet and Neumann boundary conditions relating the bulk and the boundary. It was also noted that the theory admits AdS solutions including AdS_3 solutions corresponding to topological massive supergravity theories.

3.1 Conformal constraints

For N-extended supersymmetry we consider a supermanifold M with (even|odd)-dimension $(3|2N)$. The basic structure is determined by a choice of odd tangent bundle T_1 such that the Frobenius tensor, which maps pairs of sections of T_1 to the even tangent bundle, T_0, generates the latter. We shall also suppose that there is a preferred basis $E_{\alpha I}$, $\alpha = 1, 2$; $I = 1, \ldots N$ for T_1 such that the components of the Frobenius tensor, which we shall also refer to as the dimension-zero torsion, are

$$T_{\alpha I \beta J}{}^{c} = -i\delta_{IJ}(\gamma^c)_{\alpha\beta}\,; \qquad c = 0, 1, 2\ . \tag{3.1.1}$$

At this stage T_0 is defined as the quotient, T/T_1, but we can make a definite choice for T_0 by imposing some suitable dimension one-half constraint. When this has been done, the structure group will be reduced to $SL(2, \mathbb{R}) \times SO(N)$, with the Lorentz vector indices being acted on by the local $SO(1, 2)$ associated with $SL(2,\mathbb{R})$. The dimension-zero torsion (3.1.1) is also invariant under local Weyl rescalings, although we shall not include this factor in the structure group. This indicates that we can expect to find a conformal multiplet. With respect to this structure we have preferred basis vector fields $E_{\mathcal{A}} = (E_a, E_{\underline{\alpha}}) = (E_a, E_{\alpha I})$ with dual one-forms $E^{\mathcal{A}} = (E^a, E^{\underline{\alpha}}) = (E^a, E^{\alpha I})$, the latter being related to the coordinate basis forms $dz^{\mathcal{M}} = (dx^m, d\theta^{\underline{\mu}})$ by the supervielbein matrix $E_{\mathcal{M}}{}^{\mathcal{A}}$, i.e. $E^{\mathcal{A}} = dz^{\mathcal{M}} E_{\mathcal{M}}{}^{\mathcal{A}}$. Here, coordinate indices are taken from the middle of the alphabet, preferred basis indices from the beginning, while even (odd) indices are latin and greek respectively. Underlined odd indices run from 1 to $2N$, and $SO(N)$ vector indices are denoted I, J etc.

We now introduce a set of connection one-forms, $\Omega_{\mathcal{A}}{}^{\mathcal{B}}$, for the above structure group. We have

$$\begin{aligned} \Omega_a{}^{\underline{\beta}} &= \Omega_{\underline{\alpha}}{}^b = 0 \\ \Omega_{\alpha I}{}^{\beta J} &= \delta_I{}^J \Omega_\alpha{}^\beta + \delta_\alpha{}^\beta \Omega_I{}^J \\ \Omega_a{}^b &= -(\gamma_a{}^b)_\alpha{}^\beta \Omega_\beta{}^\alpha \ . \end{aligned} \tag{3.1.2}$$

Spinor indices α, β are raised and lowered by the epsilon tensor, while Lorentz and $SO(N)$ vector indices are raised by the corresponding metrics η_{ab}, δ_{IJ}. We have $\Omega_{\alpha\beta} = \Omega_{\beta\alpha}$ while Ω_{ab} and Ω_{IJ} are antisymmetric. The torsion and curvature are defined in the usual way

$$\begin{aligned} T^{\mathcal{A}} &= DE^{\mathcal{A}} := dE^{\mathcal{A}} + E^{\mathcal{B}} \Omega_{\mathcal{B}}{}^{\mathcal{A}} \\ R_{\mathcal{A}}{}^{\mathcal{B}} &= d\Omega_{\mathcal{A}}{}^{\mathcal{B}} + \Omega_{\mathcal{A}}{}^{\mathcal{C}} \Omega_{\mathcal{C}}{}^{\mathcal{B}} \ . \end{aligned} \tag{3.1.3}$$

The Bianchi identities are

$$\begin{aligned} \mathcal{I}^{\mathcal{A}} := DT^{\mathcal{A}} - E^{\mathcal{B}} R_{\mathcal{B}}{}^{\mathcal{A}} &= 0 \\ \mathcal{I}_{\mathcal{A}}{}^{\mathcal{B}} := DR_{\mathcal{A}}{}^{\mathcal{B}} &= 0 \ . \end{aligned} \tag{3.1.4}$$

Equation (3.1.1) does not simply determine the structure group, it is also a constraint. With an appropriate choice of dimension one-half connections and of T_0, and making use of the dimension one-half Bianchi identity, one finds that all components of the dimension one-half torsion may be set to zero:

$$T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} = T_{a\underline{\beta}}{}^{c} = 0 \ . \tag{3.1.5}$$

Imposing further conventional constraints corresponding to the dimension-one connection components we find that the dimension-one torsion can be chosen to have the form

$$\begin{aligned} T_{ab}{}^{c} &= 0 \\ T_{a\beta J}{}^{\gamma K} &= (\gamma_a)_\beta{}^\gamma K_J{}^K + (\gamma^b)_\beta{}^\gamma L_{abJ}{}^K \ , \end{aligned} \tag{3.1.6}$$

where K_{IJ} is symmetric and L_{abIJ} is antisymmetric on both pairs of indices. The dimension-one curvatures are

$$\begin{aligned} R_{\alpha I\beta J,cd} &= -2i(\gamma_{cd})_{\alpha\beta} K_{IJ} - 2i\varepsilon_{\alpha\beta} L_{cdIJ} \\ R_{\alpha I\beta J,KL} &= i\varepsilon_{\alpha\beta}(M_{IJKL} + 4\delta_{[I[K}K_{J]L]}) \\ &- i(\gamma^a)_{\alpha\beta}(4\delta_{(I[K}L_{aJ)L]} - \delta_{IJ}L_{aKL}) \ , \end{aligned} \tag{3.1.7}$$

where $L_{ab} = \varepsilon_{abc}L^c$, and M_{IJKL} is totally antisymmetric. The dimension three-halves Lorentz curvature is

$$R_{a\beta J,cd} = -\frac{i}{2}(\gamma_a \Psi_{cd} - 2\gamma_{[c}\Psi_{d]a})_{\beta J} \ , \tag{3.1.8}$$

where the dimension three-halves torsion has been rewritten as $\Psi_{ab\gamma K}$; its leading component is the gravitino field strength. The $SO(N)$ curvature, $R_{a\beta J,KL}$ has a gamma-traceless part and spinor part given by

$$\begin{aligned} \hat{R}_{a\beta I,JK} &= \chi_{a\beta IJK} - i\delta_{I[J}\hat{\Psi}_{a\beta K]} \\ R_{\alpha I,JK} &= \rho_{\alpha I,JK} - 2\lambda_{\alpha IJK} + 2\delta_{I[J}\rho_{\alpha K]} \ , \end{aligned} \tag{3.1.9}$$

where we have decomposed the dual of the gravitino field strength as $\Psi_a = \hat{\Psi}_a + \gamma_a \Psi$ and where $\rho_{I,JK}$ and λ_{IJK} are in the irreducible (i.e. traceless) tableaux $\yng(2,1)$ and $\yng(1,1,1)$. The field χ is also totally antisymmetric as well as being Lorentz gamma-traceless. The derivative of K_{IJ} is given by

$$D_{\alpha I}K_{JK} = 2\rho_{\alpha(J,K)I} + 2\delta_{I(J}\kappa_{\alpha K)} + \delta_{JK}\kappa'_{\alpha I} \ , \tag{3.1.10}$$

while the derivative of L_{aIJ} is

$$D_{\alpha I}L_{aJK} = \chi_{a\alpha IJK} + i\delta_{I[J}\hat{\Psi}_{a\alpha K]} + (\gamma_a)_\alpha{}^\beta(\lambda_{IJK} + \rho_{I,JK} + 2\delta_{I[J}\sigma_{K]})_\beta \ , \tag{3.1.11}$$

The spin one-half fields in the vector representation of $SO(8)$ are related by

$$\kappa = \frac{i}{2}\Psi \ , \qquad \kappa' = 2\sigma - \frac{i}{4}\Psi \ , \qquad \rho = \sigma + \frac{i}{2}\Psi \ . \tag{3.1.12}$$

In addition, we have

$$D_{\alpha I} M_{JKLM} = i\lambda_{IJKLM} + 12i\delta_{I[J}\lambda_{\alpha KLM]} \ . \tag{3.1.13}$$

This geometry describes an off-shell superconformal multiplet [82]. The interpretation of the dimension-one fields, K, L, M, is as follows. The geometry is determined by the basic constraint (3.1.1) which is invariant under Weyl rescalings where the parameter is an unconstrained scalar superfield. This means that some of the fields that appear in the geometry do not belong to the conformal supergravity multiplet. At dimension one K and L are of this type, so that we could set them to zero if we were only interested in the superconformal multiplet. The field M_{IJKL}, on the other hand, can be considered as the field strength superfield for the conformal supergravity multiplet [82].[1] Similarly, at dimension three-halves, the fields λ_{IJK} and λ_{IJKLM} are components of the Cotton superfield, while σ, ρ and χ are like K and L in that their leading components can be removed by super-Weyl transformations. It is easy to see that these fields correspond to the θ^3 components of a scalar superfield.

The fact that M is not expressible in terms of the torsion is due to a lacuna in Dragon's theorem [84, 85] which in higher-dimensional spacetimes states that the curvature is so determined [60]. We recall that in three-dimensional spacetime there is no Weyl tensor but that its place is taken by the dimension-three Cotton tensor. This turns out to be a component of the superfield M_{IJKL} so that we could refer to the latter as the super Cotton tensor. Using the notation $[k, l]$ to denote fields that have k antisymmetrised $SO(N)$ indices and l symmetrised spinor indices, one can see that the component fields of the superconformal multiplet fall into two sequences starting from M_{IJKL}. The first has fields of the type $[4-p, p]$, where the top ($[4, 0]$) component is the supersymmetric Cotton tensor, while the second has fields of the type $(4+p, p)$ and therefore includes higher spin fields for $N > 8$. There is also a second scalar $[4, 0]$ at dimension two. Fields with two or more spinor indices obey

[1] This was discussed explicitly in for the case of $N = 8$ in [83].

covariant conservation conditions so that each field in the multiplet has two degrees of freedom multiplied by the dimension of the $SO(N)$ representation, provided that we count the dimension one and two scalars together. It is easy to see that the number of bosonic and fermionic degrees of freedom in this multiplet match. Diagrammatically, we have the following picture:

$$\begin{tikzcd}
 & & & & {[4,0]} \arrow[dl] \arrow[dr] & & & & \\
 & & & {[5,1]} \arrow[dl] \arrow[dr] & & {[3,1]} \arrow[dl] \arrow[dr] & & & \\
 & & {[6,2]} \arrow[dl] & & {[4,0]} & & {[2,2]} \arrow[dr] & & \\
 & {[7,3]} \arrow[dl] & & & & & & {[1,3]} \arrow[dr] & \\
 {} & & & & & & & & {[0,4]}
\end{tikzcd}$$

The dimension of the top field is one and thereafter increases stepwise by one-half. The spins of the fields are given by the second entry divided by two. For $N < 4$ the top field will be the one with N internal indices; for example, in $N = 3$ it will be the dimension three-halves field $[3,1]$. The right sequence clearly terminates at $[0,4]$ but the left sequences can continue to higher spin for $N > 8$. The fields $[2,2], [3,1]$ and $[0,4]$ are the $SO(N)$ gauge field strength, the supersymmetric partner of the Cotton tensor (Cottino), and the Cotton tensor respectively. In the case of $N = 6$ there is an additional $U(1)$ fields strength $[6,2]$ that plays a key rôle in the ABJM model, while in $N = 8$ one can impose a duality condition on the dimension one scalar fields that halves the multiplet; the fields in the left sequence become the duals of those in the right sequence. The dimension-two scalar fields also obey a duality constraint but it is opposite to that for the dimension-one scalars.

A consequence of this is that there is no off-shell Lagrangian for the $N = 8$ theory.

3.1.1 $N = 6$

In the case of $N = 6$, the additional $U(1)$ fields strength [6, 2] discussed in the previous section allows us to introduce a new field strength two-form G that satisfies a simple Bianchi, $dG = 0$. At dimension one we can take

$$G_{\alpha I\beta J} = i\varepsilon_{\alpha\beta}M_{IJ} \ , \tag{3.1.14}$$

where $M_{IJ} := \frac{1}{4!}\varepsilon_{IJKLMN}M^{KLMN}$ is the dual of the four-index scalar appearing in the dimension-one $SO(6)$ curvature. The dimension three-halves Bianchi identity for G then implies that

$$D_{\alpha I}M_{JK} = 2i\delta_{I[J}\lambda_{\alpha K]} + 3i\tilde{\lambda}_{\alpha IJK} \ , \tag{3.1.15}$$

where $\tilde{\lambda}_{IJK}$ is the dual of λ_{IJK} and λ_I is the dual of λ_{IJKLM}. Indeed, (3.1.15) is equivalent to (3.1.13) for $N = 6$. The dimension three-halves component of G is

$$G_{a\beta J} = -i(\gamma_a\lambda_J)_\beta \ . \tag{3.1.16}$$

3.1.2 $N = 8$

The case $N = 8$ is special for two reasons. Firstly, it is possible to impose a self-duality constraint on the superfield M_{IJKL} which reduces the size of the conformal supergravity multiplet to $128 + 128$. The fields are the graviton, 8 gravitini, the $SO(8)$ gauge fields, the dimension-one scalars M_{IJKL}, a matching dimension-two set with opposite duality and 56 dimension three-halves spinor fields (three-index antisymmetric field $\lambda_{\alpha IJK}$). The second feature is that it is possible in this case to take the R-symmetry group to be $Spin(8)$ rather than $SO(8)$. It turns out that this is the correct choice in order to describe the theories we are interested in, and so we shall switch to this for the remainder of the paper. We denote the spinor indices by $A, B, \ldots$ ((0010) representation) and $A'.B' \ldots$ ((0001) representation), while we keep $I, J, \ldots$ for the vector representation (1000). All three types of index can take 8 values. So for $N = 8$ we shall take the basis odd one-forms to be $E^{\alpha A}$, and in the above formulae replace all the internal vector indices by unprimed spinorial ones. Thus the non-zero components of the torsion are, at dimension zero

$$T_{\alpha A \beta B}{}^{c} = -i\delta_{AB}(\gamma^c)_{\alpha\beta}\,; \qquad c = 0, 1, 2 \ , \tag{3.1.17}$$

and, at dimension one,

$$T_{a\beta B}{}^{\gamma C} = (\gamma_a)_\beta{}^\gamma K_B{}^C + (\gamma^b)_\beta{}^\gamma L_{abB}{}^C \ . \tag{3.1.18}$$

The dimension-one curvatures are

$$\begin{aligned} R_{\alpha A\beta B,cd} &= -2i(\gamma_{cd})_{\alpha\beta}K_{AB} - 2i\varepsilon_{\alpha\beta}L_{cdAB} \\ R_{\alpha A\beta B,CD} &= i\varepsilon_{\alpha\beta}(M_{ABCD} + 4\delta_{[A[C}K_{B]D]}) \\ &- i(\gamma^a)_{\alpha\beta}(4\delta_{(A[C}L_{aB)D]} - \delta_{AB}L_{aCD}) \ , \end{aligned} \tag{3.1.19}$$

The field M_{ABCD} can be self- or anti-self-dual; in the former this is the representation (2000), i.e. a symmetric traceless second-rank tensor, while in the second case the representation is (0002) which is the anti-self-dual fourth-rank tensor. We shall choose the former. We denote this object by C_{IJ}, so

$$M_{ABCD} = \frac{1}{16}(\Sigma^{IK})_{[AB}(\Sigma^{J}{}_{K})_{CD]}C_{IJ} \ , \tag{3.1.20}$$

or, inverting,

$$C_{IJ} = (\Sigma^{IK})_{AB}(\Sigma^{J}{}_{K})_{CD}M^{ABCD} \ . \tag{3.1.21}$$

The $SO(8)$ dimension-one curvature is

$$\begin{aligned} R_{\alpha A\beta B,IJ} &= \frac{1}{4}(\Sigma_{IJ})^{CD}R_{\alpha A\beta B,CD} \\ & i\varepsilon_{\alpha\beta}(\frac{1}{4}(\Sigma_{[I}{}^{K})_{AB}C_{J]K} + (\Sigma_{IJ})_{[A}{}^{C}K_{B]C}) \\ & +\frac{i}{2}(\delta_{AB}L_{aIJ} + \frac{1}{2}(\Sigma_{IJKL})_{AB}L_{a}{}^{KL}) \ , \end{aligned} \tag{3.1.22}$$

where $L_{aIJ} = \frac{1}{4}(\Sigma_{IJ})^{CD}L_{aCD}$. The curvature in the primed spinor representation is

$$R_{\alpha A\beta B,C'D'} = \frac{1}{4}(\Sigma^{IJ})_{C'D'}R_{\alpha A\beta B,IJ} \tag{3.1.23}$$

The dimension three-halves curvatures and relations are given by equations (3.1.9) to (3.1.13) but with $I, J, K, ..$ replaced by A, B, C. The five-index λ spinor is the dual of λ_{ABC} multiplied by a factor of 1/3. The field λ_{ABC} can also be written as a Σ-traceless primed vector-spinor $\lambda_{IA'}$, with

$$\lambda_{IA'} = -\frac{1}{3}(\Sigma^{J})_{A'}{}^{A}(\Sigma_{IJ})^{BC}\lambda_{ABC} \ , \tag{3.1.24}$$

and it is easy to check that

$$\lambda_{\alpha IA'} = \frac{i}{15}(\Sigma^{J})_{A'}{}^{A}D_{\alpha A}C_{IJ} \ . \tag{3.1.25}$$

3.2 Matter multiplets

In the superspace approach to superconformal models we consider scalar multiplets and vector multiplets as well as the conformal supergravity multiplet. We shall take both the scalar and vector multiplets to be on-shell, although there is an off-shell version of the conformal $N = 6$ vector multiplet in harmonic superspace. For the vector multiplets the conformal action is the Chern-Simons term so that in the absence of any scalar multiplets the equation of motion states that the spacetime field strength is zero, and the superspace extension of this is simply that the whole of the superspace field strength vanishes. When matter is present the spacetime field strength will be given by the dual of the matter current and in superspace this implies that all components of the field strength will be given as bilinears in the matter fields.

The programme is therefore to write down the Ricci identity for the scalar fields, which is equivalent to demanding that the supersymmetry algebra closes on them, and the Bianchi identities for the gauge fields and the geometry and to ensure that they are mutually compatible and consistent when the relevant constraints are imposed. In this section we shall briefly recap what happens in flat superspace in a Lie-algebra, which was developped in the BLG case in . We shall take the gauge group to be $G \times G'$ in $N = 6$ and recover the result that $G = G' = SU(2)$ for $N = 8$.

3.2.1 N=6

In the ABJM case the scalar field Z_A is complex, in the four-dimensional spinor representation of $SU(4)$, the spin group of $SO(6)$. As it is complex it can also carry a $U(1)$ charge q with respect to the additional $U(1)$ R-symmetry factor. In flat superspace the basic equations are: the variation of the scalar,

$$D_{\alpha I} Z_A = i(\Sigma_I)_{AB} \Lambda^B_\alpha \ . \tag{3.2.1}$$

the Ricci identity,

$$[D_{\alpha I}, D_{\beta J}] Z_A = i\delta_{IJ}(\gamma^a)_{\alpha\beta} D_a Z_A - F_{\alpha I \beta J} Z_A + Z_A F'_{\alpha I \beta J} \ , \tag{3.2.2}$$

the variation of Λ,

$$D_{\alpha I} \Lambda^B_\beta = \frac{1}{2}(\gamma^a)_{\alpha\beta}(\Sigma_I)^{BC} D_a Z_C + \varepsilon_{\alpha\beta} H_I{}^B + (\gamma^a)_{\alpha\beta} H_{aI}{}^B \ , \tag{3.2.3}$$

and the dimension-one components of the gauge field strengths,

$$F_{\alpha I \beta J} = ia\varepsilon_{\alpha\beta}\, Z\Sigma^{IJ} Z^* \qquad \text{and} \qquad F'_{\alpha I \beta J} = ia'\varepsilon_{\alpha\beta}\, Z^*\Sigma^{IJ} Z \ . \tag{3.2.4}$$

As in the the BLG case we can substitute (3.2.3) in (3.2.2) to find, firstly, that $H_{aI}{}^B = 0$, and then that

$$4\Sigma_{[I} H_{J]} = b(Z\Sigma^{IJ} Z^* Z - ZZ^*\Sigma^{IJ} Z) + c(Z\Sigma^{IJ} Z^* Z + ZZ^*\Sigma^{IJ} Z) \ . \tag{3.2.5}$$

Making the $SU(4)$ structure of these terms more explicit we find that the b term is

$$2b(\Sigma_{IJ})^B{}_C Z_{(A} \bar{Z}^C Z_{B)} \ , \tag{3.2.6}$$

whereas the c term is antisymmetric on AB. (3.2.6) can be rewritten as

$$2b(\Sigma^{KLM}\Sigma_{IJ})_{AB} \xi_{KLM}{}^B \ . \tag{3.2.7}$$

The spinor ξ can be expanded in terms of irreducible representations and we find that it contains the thirty-six dimensional (201) representation that does not drop out of (3.2.5) and that cannot be absorbed in H_I. So we again have to choose $b = 0$. On the other hand, the c term is compatible with (3.2.5) for any choice of gauge group of the form $G \times G'$. One finds

$$H_I{}^A = c(\zeta_I - \frac{1}{4}\Sigma_I\Sigma^J\zeta_J)^A \ , \tag{3.2.8}$$

where

$$\zeta_I{}^A := -\frac{1}{2}(\Sigma_I)^{BC}Z_B\bar{Z}^A Z_C \ . \tag{3.2.9}$$

3.2.2 N=8

For $N = 8$ the scalar multiplet has eight scalars X_I and eight spinors $\Lambda_{\alpha A'}$, so the constraint on the superfield X_I must be

$$D_{\alpha A}X_I = i(\Sigma_I)_{AA'}\Lambda_\alpha^{A'} \ . \tag{3.2.10}$$

Here, the derivative is gauge-covariant with respect to the group $G \times G'$, so the Ricci identity is

$$[D_{\alpha A}, D_{\beta B}]X_I = i\delta_{AB}(\gamma^a)_{\alpha\beta}D_a X_I - F_{\alpha A\beta B}X_I + X_I F'_{\alpha A\beta B} \ . \tag{3.2.11}$$

To check the consistency of the Ricci identity we can parametrise the variation of Λ by

$$D_{\alpha A}\Lambda_{\beta B'} = \frac{1}{2}(\gamma^a)_{\alpha\beta}(\Sigma_I)_{AB'}D_a X_I + \varepsilon_{\alpha\beta}H_{AB'} + (\gamma^a)_{\alpha\beta}H_{aAB'} \ . \tag{3.2.12}$$

The first term is there in the absence of interactions while the fields appearing the second and third will be functions of the matter fields as we are on-shell. In order to determine these we need first to say something about the lowest-dimensional components of the fields strengths. The scalar X has dimension one-half, Λ dimension one and $F_{0,2}$ also has dimension one. The latter can therefore only be bilinear in X. We have

$$F_{\alpha A\beta B} = ia\varepsilon_{\alpha\beta}(\Sigma^{IJ})_{AB}X_I X_J^* \qquad \text{and} \qquad F'_{\alpha A\beta B} = ia'\varepsilon_{\alpha\beta}(\Sigma^{IJ})_{AB}X_I^* X_J \ , \tag{3.2.13}$$

where a, a' are real constants. Using (3.2.10) we can easily see that these constraints are compatible with the Bianchi identities for F, F'. The dimension three-halves component of F is given by

$$F_{a\beta B} = (\gamma_a \chi)_{\beta B} \ , \tag{3.2.14}$$

and similarly for F' where

$$\begin{aligned} \chi_{\alpha A} &= ia(\Sigma^I(X_I\Lambda^* - \Lambda X_I^*))_{\alpha A} \\ \chi'_{\alpha A} &= ia'(\Sigma^I(X_I^*\Lambda - \Lambda^* X_I))_{\alpha A} \ . \end{aligned} \tag{3.2.15}$$

Furthermore, using the identity for the Bianchi identity $D\mathcal{I} = 0$, we find that the $(1,2)$ component of $\mathcal{I}$ determines the $(2,0)$ component of F and that the higher-dimensional components are then automatically satisfied.

Substituting (3.2.12) into (3.2.11) we find, in the case of flat superspace, that $H_{aBC'} = 0$, and that

$$\begin{aligned} 2(\Sigma_I)_{[A}{}^{C'} H_{B]C'} &= (\Sigma^{JK})_{AB}(b(X_J X_K^* X_I - X_I X_J^* X_K) \\ &+ c(X_J X_K^* X_I + X_I X_J^* X_K)) \ , \end{aligned} \tag{3.2.16}$$

where $b := \frac{1}{2}(a + a')$ and $c := \frac{1}{2}(a - a')$. The terms cubic in X contain the representations (1000), (0011) and (1100) of $SO(8)$, but only the first two are contained in $H_{AB'}$. The mixed symmetry representation must therefore be excluded. It appears inevitably in the b term, so that b must be set to zero, or $a = -a'$. This means that the coefficients of the two Chern-Simons terms in the space-time action must have equal magnitude and opposite sign. The c term will also have a mixed symmetry component except for the case $SU(2) \times SU(2)$, and when X_I is real,

$$X_x{}^{x'} \to \bar{X}^x{}_{x'} = \varepsilon^{xy} X_y{}^{y'} \varepsilon_{y'x'} \ \forall \ I \ , \tag{3.2.17}$$

where $x, x' = 1, 2$ are doublet indices for the two $SU(2)$s. In this case, it is easy to see, using the cyclic formula

$$AB^*C + AC^*B = A\mathrm{tr}(B^*C) \ , \tag{3.2.18}$$

valid for any real fields (as in (3.2.17)) in the bi-fundamental representation of $SU(2) \times SU(2)$. In this case one finds that only the totally antisymmetric X^3 term survives and that

$$H_{AA'} = \frac{c}{6}(\Sigma^{IJK})_{AA'} X^3_{IJK} \ , \tag{3.2.19}$$

where $X^3_{IJK} := X_{[I} X_J^* X_{K]}$.

3.3 Coupling to supergravity

In this section we consider the coupling of the matter-gauge systems to conformal supergravity. The idea is that we have to satisfy the Bianchi identities in the gravity and gauge sectors and the Ricci identity for the matter fields. For the ABJM case it turns out that the parameter b must still be set equal to zero and that the scalar multiplet can be coupled to the off-shell superconformal geometry, while for the BLG case the situation is more complicated. There one can couple the scalar multiplet to on-shell conformal supergravity, but only if the parameter b is non-zero.

3.3.1 N=6

The basic constraint on the scalar multiplet (3.2.1) is unchanged (although the derivative now includes the geometrical connections), and the dimension-one components of the gauge field strength tensors are also unaltered. However, the Ricci identity (3.2.2) is amended to

$$\begin{aligned}[][D_{\alpha I}, D_{\beta J}]Z_A &= i\delta_{IJ}(\gamma^a)_{\alpha\beta}D_a Z_A - F_{\alpha I\beta J}Z_A + Z_A F'_{\alpha I\beta J} \\ &- iqG_{\alpha I\beta J}Z_A - R_{\alpha I\beta J,A}{}^B Z_B \ , \end{aligned} \tag{3.3.1}$$

where q is the $U(1)$ charge of the scalar field Z and the last term involves the $SO(6)$ curvature in the spin representation, $R_A{}^B = \frac{1}{4}(\Sigma^{IJ})_A{}^B R_{IJ}$. The variation of Λ is still given by (3.2.3), although the field H_a cannot be set to zero. We now want to investigate the Ricci identity on the scalar fields using the variation of the spinor, as before. The crucial terms come from the $M \times Z$ terms in the $SO(6)$ and $U(1)$ curvatures acting on Z. The key point is that these give rise to a composite object $M_{IJ}Z_A$ which decomposes into (201) + (011) + (100). The matter multiplet, as we discussed above, does contribute a term in the (201) when $b \neq 0$, as can be seen from (3.2.6). However, M_{IJ} can only be proportional to $\mathrm{tr}(Z\Sigma_{IJ}Z^*)$ on-shell, whereas (3.2.6) cannot be written in this form for any choice of gauge group $G \times G'$, not even for $G = G' = SU(2)$. In this case one can use (3.2.18) to write (3.2.6) as a some of two terms involving traces. One of these has the correct form to be absorbed by $M_{IJ}Z_A$, but the other has the form $tr(ZZ)Z^*$ which cannot. We therefore conclude that $b = 0$ in the presence of conformal supergravity as well as in flat space.

This result means that the (201) representation in $M_{IJ}Z$ cannot be absorbed by the matter sector and hence the two MZ terms in (3.3.1) must be arranged so that this term cancels between them. This requires the charge q to be $-\frac{1}{2}$. With this choice the supersymmetry algebra on the scalars closes provided that the H-functions in the variation of Λ are chose to be

$$\begin{aligned}
H_{aI}{}^A &= L_{aIJ}(\Sigma^J Z)^A - \frac{1}{2}L_a{}^{JK}(\Sigma_{IJK}Z)^A \\
H_I{}^A &= \frac{1}{2}K_{IJ}(\Sigma^J Z)^A \\
&\quad +c(\zeta_I - \frac{1}{4}\Sigma_I\Sigma^J\zeta_J)^A - i\mu_I{}^A + \frac{3i}{8}(\Sigma_I\Sigma^J\mu_J)^A \ , \qquad (3.3.2)
\end{aligned}$$

where

$$M_{IJ}Z := \hat{\mu}_{IJ} + \Sigma_{[I}\mu_{j]} \ . \qquad (3.3.3)$$

Here, $\hat{\mu}_{IJ}$ denotes the (201), while μ_I is the some of the other two representations, (011) and (100).

3.3.2 N=8

The situation is somewhat different in $N = 8$. We shall start with the closure of supersymmetry on the scalars, for which (3.2.11) is modified to

$$[D_{\alpha A}, D_{\beta B}]X_I = i\delta_{AB}(\gamma^a)_{\alpha\beta}D_aX_I - F_{\alpha A\beta B}X_I + X_IF'_{\alpha A\beta B} - R_{\alpha A\beta B,I}{}^JX_J \ , \tag{3.3.4}$$

while the variation of the fermion is given in (3.2.12). We now find that there is a solution for the H-fields in the variation of Λ for the gauge group $SU(2) \times SU(2)$ with non-vanishing conformal supergravity provided that $b \neq 0$. Explicitly we find

$$\begin{aligned} H_{AA'} &= \frac{1}{2}(\Sigma^I)_{A'}{}^BK_{AB}X_I + \frac{b}{2}(\Sigma^I)_{AA'}(\mathrm{tr}(X_IX_J^*)X_J - \frac{1}{4}\mathrm{tr}X^2X_I) \\ &\quad -\frac{c}{6}X^3_{IJK} \\ H_{aAA'} &= \frac{1}{4}(-(\Sigma^I)_{AA'}L_{aIJ}X_J + \frac{1}{2}(\Sigma^{IJK})_{AA'}L_{aIJ}X_K) \ , \end{aligned} \tag{3.3.5}$$

where $\mathrm{tr}X^2 := \mathrm{tr}(X_IX_I^*)$. The b term in $H_{AA'}$ can be absorbed by the geometry provided that we choose

$$C_{IJ} = 8b(\mathrm{tr}(X_IX_J^*) - \frac{1}{8}\delta_{IJ}\mathrm{tr}X^2) \ . \tag{3.3.6}$$

It might be thought that the terms in (3.3.5) involving K_{AB} and L_{aBC} could be ignored since their leading components can be gauged away, but this is not correct because the spinorial derivatives of these fields include terms involving the gravitino field strength and the field λ_{ABC}. In particular, the latter turns out to be (using (3.1.25))

$$\lambda_I = -\frac{14ib}{3}(\mathrm{tr}(X_I\Lambda^*) - \frac{1}{7}\Sigma_{IJ}\mathrm{tr}(X_J\Lambda^*)\ . \tag{3.3.7}$$

This essentially solves the problem in superspace. It is tedious, but straightforward, to verify that the supersymmetry algebra closes on Λ and to obtain as a bi-product the equation of motion for the spinor field. It is

$$\begin{aligned}\gamma^a D_a\Lambda &= -\frac{b}{4}(2\Sigma^{IJ}\mathrm{tr}(\Lambda X_I)X_J + 10\mathrm{tr}(\Lambda X_I)X_I - \mathrm{tr}X^2\Lambda)\\ &\quad +c\Sigma^{IJ}X_I\Lambda X_J - \frac{3}{4}\Sigma^I\Psi X_I + \frac{1}{4}L_{aIJ}\gamma^a\Sigma^{IJ}\Lambda\end{aligned} \tag{3.3.8}$$

In [63] it was shown that this equation indeed transforms homogeneously under super-Weyl transformation. From this one finds the pure scalar terms in the scalar equation of motion to be

$$\begin{aligned}D^aD_aX_I &= b^2(3\mathrm{tr}(X_IX_J)\mathrm{tr}(X_JX_K)X_K - \mathrm{tr}X^2\mathrm{tr}(X_IX_J)X_J\\ &\quad -\frac{1}{2}\mathrm{tr}(X_JX_K)\mathrm{tr}(X_JX_K)X_I + \frac{3}{16}(\mathrm{tr}X^2)^2X_I)\\ &\quad +2c^2X_JX^3_{IJK}X_K\ .\end{aligned} \tag{3.3.9}$$

The last term can be rewritten in terms of traces if desired. This corresponds to a potential that is proportional to

$$\begin{aligned}V(X) &\propto b^2(\frac{1}{2}\mathrm{tr}(X_IX_J)\,\mathrm{tr}(X_IX_K)\,\mathrm{tr}(X_JX_K) - \frac{1}{4}\mathrm{tr}X^2\,\mathrm{tr}(X_IX_J)\,\mathrm{tr}(X_IX_J)\\ &\quad +\frac{1}{48}(\mathrm{tr}X^2)^3) - c^2\mathrm{tr}(X^3_{IJK}X^3_{IJK})\ .\end{aligned} \tag{3.3.10}$$

For the geometrical sector the general analysis given previously shows that we have a complete solution of the Bianchi identities provided that the field M_{ABCD} satisfies (2.13) (with the indices replaced by A, B etc and with λ_5 being the dual of λ_3). That this is so is easily verified from (3.3.6) and (3.2.10). For the gauge sector we have already shown that the Bianchi identities are satisfied given (3.2.13) and (3.2.10). The dimension three-halves components are given by (3.2.14) and (3.2.15). The dimension-two components are given by

$$\begin{aligned} F_{ab} &= -a\varepsilon_{ab}{}^{c}(X_I D_c X_I^* - D_c X_I X_I^* + \frac{i}{2}\Lambda_{A'}\gamma_c\Lambda^*_{A'}) \\ F'_{ab} &= -a'\varepsilon_{ab}{}^{c}(X_I^* D_c X_I - D_c X_I^* X_I + \frac{i}{2}\Lambda^*_{A'}\gamma_c\Lambda_{A'}) \ . \end{aligned} \tag{3.3.11}$$

3.4 $N = 8$ models with $SO(n)$ gauge groups

In the $N = 8$ modification of BLG we have seen that setting $b = 0$ immediately implies that the background has to be superconformally flat, because C_{IJ} and hence all of the field strengths in the super Cotton tensor must vanish. On the other hand one can set $c = 0$ without getting a free model. If the gauge group is $SU(2) \times SU(2)$ this means that the two Chern-Simons terms in the space-time Lagrangian have equal magnitudes and signs. They can therefore be rewritten as a single $SO(4)$ Chern-Simons term. It turns out, as we shall now show, that this model can be generalised to an $SO(n)$ gauge group.

We now take the scalar field $X_I^r, r = 1, \ldots n$ to transform under the vector representation of $SO(n)$ as well as $SO(8)$. Then (3.3.4) becomes

$$[D_{\alpha A}, D_{\beta B}]X_I^r = i\delta_{AB}(\gamma^a)_{\alpha\beta}D_a X_I - F^{rs}_{\alpha A\beta B}X_I^s - R_{\alpha A\beta B,I}{}^{J}X_J^r \ , \tag{3.4.1}$$

while in (3.2.11) both X and Λ carry an extra $SO(n)$ vector index and (3.2.13) is replaced by

$$F^{rs}_{\alpha A\beta B} = ia(\Sigma^{IJ})_{AB} X^r_I X^s_J \ . \tag{3.4.2}$$

We then find that the supersymmetry algebra closes on the scalars if we take

$$C_{IJ} = 8a(X^2_{IJ} - \frac{1}{8}\delta_{IJ} X^2_{KK}) \tag{3.4.3}$$

and

$$H_{AA'} = \frac{1}{2}(\Sigma^I)_{A'}{}^B K_{AB} X_I + \frac{a}{2}(X^2_{IJ} X_J - \frac{1}{4}\delta_{IJ} X^2) \ . \tag{3.4.4}$$

Chapter 4

On-shell Poincaré supergravity

It has been known for many years that maximal supergravity theories have hidden rigid symmetry groups that increase in dimension as the dimension of spacetime decreases [18]. Maximal supergravity in three dimensions is special in the sense that the symmetry group E_8 is the largest finite one in the E series; in $D = 2$ one has E_9 [64] and more recently it has been suggested that these symmetries might be extended to E_{10} [65] or E_{11} [38]. As mentioned in the introduction, the 128+128 on-shell degrees of freedom of maximal supergravity in three dimensions [66] are entirely non-gravitational, so that in a sense the theory is really an $SO(16)\backslash E_8$ non-linear sigma model, although there is an induced geometrical structure.

There is also an interesting set of theories that have half-maximal supersymmetry. In $D = 3$ there are a number of half-maximal supergravity theories with sigma models of the form $(SO(8)\times SO(n))\backslash SO(8,n)$ that were also first introduced in [66] and further studied in [49].

In the following chapter we study maximal and half-maximal theories in a superspace setting, starting, from the off-shell superconformal geometry discussed in the previous chapter. We will do this by introducing the appropriate sigma model in the context of this supergravity background and show how the latter can accommodate it by making the necessary identifications. For the maximal case we analyse the geometrical Bianchi identities up to dimension two. This allows us to identify all components of the torsion and curvature tensors and to verify explicitly that the constraints are consistent. It is important to do this as the initial set of constraints include conventional ones for the $SO(16)$ connection, whereas the sigma model fixes this connection in terms of the physical fields, so that one needs to check that these choices are compatible. For the half-maximal case we only give this discussion at dimension one. The main reason for us to introduce the super geometry of half-maximal supergravity is because we will use it to study Borcherds algebras and α'-corrections later on, hence the geometry is not our primary interest. However there are no reasons to expect that any inconsistencies would occur at dimension three-halves or two. For similar reasons we give a brief overview of type IIA and IIB supergravity at the end of this chapter.

We also refer to appendix C where the geometry of off-shell $N = 16$ supergravity in one dimension is given. We take the opportunity present this work here as it has not been given in the literature previously.

4.1 $N = 16$

The geometrical set up is that of the previous chapter. For the case of maximal supergravity we consider a supermanifold M with (even|odd)-dimension (3|32), and the following set of connection one-forms, $\Omega_A{}^B$

$$\begin{aligned} \Omega_a{}^{\underline{\beta}} &= \Omega_{\underline{\alpha}}{}^b = 0 \\ \Omega_{\alpha i}{}^{\beta j} &= \delta_i{}^j \Omega_\alpha{}^\beta + \delta_\alpha{}^\beta \Omega_i{}^j \\ \Omega_a{}^b &= -(\gamma_a{}^b)_\alpha{}^\beta \Omega_\beta{}^\alpha \, . \end{aligned} \tag{4.1.1}$$

where $\alpha = 1, 2$; $i = 1, \ldots 16$. The $SO(16)$[1] vector indices $i, j...$ are raised using δ_{ij}. We have $\Omega_{\alpha\beta} = \Omega_{\beta\alpha}$ while Ω_{ab} and Ω_{ij} are antisymmetric. Explicitly the dimensions one constraint on the torsion reads

$$T_{\alpha i \beta j}{}^{c} = -i\delta_{ij}(\gamma^c)_{\alpha\beta}\,; \qquad c = 0, 1, 2\ . \tag{4.1.2}$$

The structure group contains an $SO(16)$ factor which is associated with an $SO(16)$ principal bundle. We can introduce the sigma model via the requirement that this bundle can be lifted to a flat E_8 bundle. Our conventions for the Lie algebra, $\mathfrak{e}_8$, are as follows: the generators are (M_{ij}, N_I) where $M_{ij} = -M_{ji}$ are the generators for $\mathfrak{so}(16)$ and the remaining generators, N_I, $I = 1, \ldots 128$, transform under one of the two Weyl spinor representations of $\mathfrak{spin}(16)$. We shall denote the other representation by primed indices, e.g. I'. The algebra of $\mathfrak{e}_8$ is

$$\begin{aligned}
[M_{ij}, M^{kl}] &= -4\delta_{[i}{}^{[k} M_{j]}{}^{l]} \\
[M_{ij}, N_I] &= -\frac{1}{2}(\Sigma_{ij})_{IJ} N_J \\
[N_I, N_J] &= (\Sigma^{ij})_{IJ} M_{ij}\ ,
\end{aligned} \tag{4.1.3}$$

where the $SO(16)$ sigma matrices are denoted by Σ (see appendix for conventions).

The sigma model field can be viewed as a section $\mathcal{V}$ of the E_8 bundle. It is acted on to the right by E_8 and to the left by the local $SO(16)$ and therefore corresponds to an $SO(16)\backslash E_8$ sigma model superfield. The Maurer-Cartan form is

$$\Phi := d\mathcal{V}\mathcal{V}^{-1} := P + Q\ , \tag{4.1.4}$$

[1]Our $SO(16)$ conventions can be found in appendix A

where $Q = \frac{1}{2}\Omega^{ij}M_{ij}$ and where P takes its values in the quotient algebra, i.e. $P = P^I N_I$. From the Maurer-Cartan equation (vanishing E_8 curvature), $d\Phi + \Phi^2 = 0$, we find

$$\begin{aligned} DP &= 0 &\quad (4.1.5)\\ R &= -P^2 \ , &\quad (4.1.6)\end{aligned}$$

where $R := \frac{1}{2}R^{ij}M_{ij}$ is the $SO(16)$ curvature, while D is the $SO(16)$-covariant exterior derivative. In indices, the above equations are

$$\begin{aligned} 2D_{[A}P_{B]} + T_{AB}{}^{C}P_C &= 0 &\quad (4.1.7)\\ R_{AB,ij} &= 2P_A\Sigma_{ij}P_B \ . &\quad (4.1.8)\end{aligned}$$

We shall need to impose a constraint on the dimension one-half component of P to ensure that we have the correct number of degrees of freedom, namely 128 bosonic and fermionic. We therefore set

$$P_{\alpha i I} = i(\Sigma_i\Lambda_\alpha)_I \ , \tag{4.1.9}$$

where $\Lambda_{\alpha I'}$ describes the physical 128 spin one-half fields. The dimension-one component of (4.1.8) is then satisfied if

$$D_{\alpha i}\Lambda_{\beta I'} = \frac{1}{2}(\gamma^a)_{\alpha\beta}(\Sigma_i P_a)_{I'} \ . \tag{4.1.10}$$

We can think of P_{aI} as essentially the spacetime derivative of the physical scalar fields. In order to see this more explicitly, it is perhaps useful to look at the linearised limit. In the physical gauge we can put $\mathcal{V} = \exp(\phi^I N_I)$

where ϕ^I denotes the 128 scalars. If we now keep only terms linear in the fields we find

$$\begin{aligned} D_{\alpha i}\phi_I &= i(\Sigma_i\Lambda_\alpha)_I \\ D_{\alpha i}\Lambda_{\beta J'} &= \frac{1}{2}(\gamma^a)_{\alpha\beta}(\Sigma_i P_a)_{J'} = \frac{1}{2}(\gamma^a)_{\alpha\beta}(\Sigma_i\partial_a\phi)_{J'} \ , \end{aligned} \qquad (4.1.11)$$

where $D_{\alpha i}$ is now the usual supercovariant derivative in flat superspace. It follows from (4.1.11) that both ϕ_I and $\Lambda_{\alpha I'}$ satisfy free field equations of motion.

Note that we have now specified the $SO(16)$ connection in two ways, by choosing corresponding conventional constraints on the torsion, and explicitly in terms of $\mathcal{V}$. We therefore need to verify that these are compatible. We can easily see that they are by making use of the dimension-one component of (4.1.8). Comparing with (3.1.6), (3.1.7) we find agreement provided that

$$\begin{aligned} K_{ij} &= -\frac{i}{2}\delta_{ij}B := -\frac{i}{2}\delta_{ij}\Lambda\Lambda \\ L_{aij} &= iA_{aij} := i\Lambda\gamma_a\Sigma_{ij}\Lambda \\ M_{ijkl} &= -iB_{ijkl} := -i\Lambda\Sigma_{ijkl}\Lambda \ , \end{aligned} \qquad (4.1.12)$$

where, on the right-hand-side, the spacetime and internal spinor indices are contracted in the natural way (see appendix). The non-zero dimension-one torsion therefore becomes

$$T_{a\beta j\gamma k} = -\frac{i}{2}(\gamma_a)_{\beta\gamma}\delta_{jk}B - i(\gamma_a{}^b)_{\beta\gamma}A_{bjk} \ . \qquad (4.1.13)$$

With this, we now have a solution to the coupled Maurer-Cartan equations and geometrical Bianchi identities up to dimension one expressed entirely in terms of the physical fields. In terms of the sigma model fields the dimension-one curvatures are

$$
\begin{aligned}
R_{\alpha i \beta j, cd} &= -\delta_{ij}(\gamma_{cd})_{\alpha\beta}B + 2\varepsilon_{\alpha\beta}A_{cdij} \;\Rightarrow \\
R_{\alpha i,\beta j,\gamma\delta} &= \frac{1}{2}\delta_{ij}(\gamma^a)_{\alpha\beta}(\gamma_a)_{\gamma\delta}B - \varepsilon_{\alpha\beta}(\gamma^a)_{\gamma\delta}A_{aij} \;, \\
R_{\alpha i\beta j,kl} &= \varepsilon_{\alpha\beta}(B_{ijkl} + 2\delta_{[i[k}\delta_{j]l]}B) \\
&\quad +(\gamma^a)_{\alpha\beta}(4\delta_{(i[k}A_{aj)l]} - \delta_{ij}A_{akl}) \;. \qquad (4.1.14)
\end{aligned}
$$

Note that there is an interesting feature of this solution that does not occur in higher-dimensional maximal supergravity theories (except for $D = 11$), namely the fact that the dimension one-half torsion tensor is zero. This is easily understood in terms of group representations because in $D = 3$ the spinor field transforms as a spinor under the internal symmetry group whereas the geometrical tensors can only accommodate tensor representations. If we move up to $N = 8$ supergravity in $D = 4$, for example, the internal symmetry group is $SU(8)$ and the spin one-half fermions transform under the 56-dimensional representation. They can therefore be accommodated in the dimension one-half torsion as follows [86]:

$$
T_{\alpha i,\beta j,\dot{\gamma}k} = \varepsilon_{\alpha\beta}\bar{\Lambda}_{\dot{\gamma}ijk} \;, \qquad (4.1.15)
$$

where we have used two-component spinor notation, where $i, j, k = 1, \ldots 8$ and where $\bar{\Lambda}_{\dot{\gamma}ijk}$ is totally antisymmetric on its internal indices. Its leading component in a θ-expansion is the physical spin one-half fields in the 56 of $SU(8)$.

4.1.1 Dimension three-halves

In this section we shall check the various identities up at dimension three-halves. This will enable us to confirm the consistency of the solution and also to compute the dimension three-halves and two components of the torsion and curvature. As expected, these turn out to be functions of the physical fields, there being no gravitational degrees of freedom in three dimensions.

There are two relevant Bianchi identities, as well as the dimension three-halves components of $DP = 0$ and $R = -P^2$. They are

$$\begin{aligned}
2R_{[a\underline{\alpha},b]c} &= -T_{ab}{}^{\underline{\beta}}T_{\underline{\alpha}\underline{\beta}c} && (4.1.16)\\
2R_{a(\underline{\alpha},\underline{\beta})\underline{\gamma}} &= -2D_{(\underline{\alpha}}T_{a\underline{\beta})\underline{\gamma}} - T_{\underline{\alpha}\underline{\beta}}{}^{b}T_{ab\underline{\gamma}} && (4.1.17)\\
D_aP_{\underline{\beta}} - D_{\underline{\beta}}P_a + T_{a\underline{\beta}}{}^{\underline{\gamma}}P_{\underline{\gamma}} &= 0 && (4.1.18)\\
R_{a\beta j,kl} &= +2iP_a\Sigma_{kl}\Sigma_j\Lambda_\alpha\ . && (4.1.19)
\end{aligned}$$

Equation (4.1.16) allows us to solve for the dimension three-halves Lorentz curvature in terms of the dimension three-halves torsion. The $\theta = 0$ component of the latter can be identified as the gravitino field strength, so we shall give it a new notation $T_{ab}{}^{\underline{\gamma}} := \Psi_{ab}{}^{\underline{\gamma}} = \varepsilon_{abc}\Psi^{c\,\underline{\gamma}}$. From (3.3.2) we find that

$$\Psi_{ai} = 2\gamma^b\gamma_aP_b\Sigma_i\Lambda\ , \tag{4.1.20}$$

confirming that the gravitino field strength is completely determined by the matter fields, as promised. The dimension-three-halves Lorentz curvature is

$$R_{a\beta j,b} = i(\gamma_a\Psi_{bi} - \frac{1}{2}\eta_{ab}\gamma^c\Psi_{ci})_\beta\ . \tag{4.1.21}$$

From (4.1.18) we can determine the supersymmetry variation of P_a,

$$D_{\alpha i} P_a = i(\Sigma_i D_a \Lambda_\alpha) + iT_{a\alpha i}{}^{\beta j}(\Sigma_j \Lambda_\beta) \ , \tag{4.1.22}$$

where the second term on the right-hand-side gives terms that are cubic in Λ.

At this stage we see that we have determined the geometric tensors in terms of the physical fields, but there are several other equalities that arise of which one is the equation of motion for Λ. We can study these equalities by computing two spinorial covariant derivatives acting on Λ. We have

$$D_{\underline{\alpha}} D_{\underline{\beta}} = D^2_{\underline{\alpha}\underline{\beta}} + \frac{1}{2}\{D_{\underline{\alpha}}, D_{\underline{\beta}}\} \ . \tag{4.1.23}$$

The second-order derivative can be decomposed as

$$D^2_{\underline{\alpha}\underline{\beta}} := \varepsilon_{\alpha\beta} D^2_{ij} + (\gamma^a)_{\alpha\beta} D^2_{aij} \tag{4.1.24}$$

where the Lorentz scalar and vector terms are respectively symmetric and antisymmetric on ij. We can now evaluate two spinorial derivatives on Λ using this and the Ricci identity that enables us to express the anti-commutator in terms of the torsion times a single derivative and the curvature. We find

$$
\begin{aligned}
\varepsilon_{\alpha\beta}D^2_{ij}\Lambda_\gamma+(\gamma^a)_{\alpha\beta}D^2_{aij}\Lambda_\gamma \quad = \quad & -\frac{i}{2}\delta_{ij}(\gamma^a)_{\alpha\beta}D_a\Lambda_\gamma+\frac{i}{2}(\gamma^a)_{\beta\gamma}(\Sigma_j\Sigma_i D_a\Lambda_\alpha) \\
& +\frac{i}{2}(\gamma^a)_{\beta\gamma}T_{a\alpha i}{}^{\delta l}(\Sigma_j\Sigma_l\Lambda_\delta) \\
& -\frac{1}{2}\varepsilon_{(\alpha|\gamma|}\Lambda_{\beta)}B-\frac{1}{2}\varepsilon_{\alpha\beta}(\gamma^a\Lambda)_\gamma A_{aij} \\
& +\frac{1}{8}(B_{ijkl}+2\delta_{i[k}\delta_{l]j}B)(\Sigma^{kl}\Lambda_\gamma) \\
& +\frac{1}{8}(\gamma^a)_{\alpha\beta}(4\delta_{(i[k}A_{aj)l]} \\
& -\delta_{ij}A_{akl})(\Sigma^{kl}\Lambda_\gamma)\ . \qquad (4.1.25)
\end{aligned}
$$

We now split this equation into four parts according to the symmetries of the pairs of spinor and internal indices. Consider first the part that is symmetric on $\alpha\beta$ and on ij. After contracting the expression with $(\gamma_a)^{\alpha\beta}$ and with a little algebra one can show that this is proportional to δ_{ij}. One finds

$$
2i\gamma_a\gamma^b D_b\Lambda-\gamma_a B\Lambda+\Sigma^{ij}A_{aij}\Lambda=0\ , \qquad (4.1.26)
$$

Contracting this with γ^a we obtain the Dirac equation for Λ,

$$
\gamma^a D_a\Lambda=-\frac{i}{2}B\Lambda+\frac{i}{6}\Sigma^{ij}A_{aij}\gamma^a\Lambda. \qquad (4.1.27)
$$

The gamma-traceless part of (4.1.26) must therefore vanish identically. That it does so is due to the identity

$$
\Sigma^{ij}\Lambda_{(\alpha}\Lambda_\beta\Sigma_{ij}\Lambda_{\gamma)}=0\ . \qquad (4.1.28)
$$

The part that is symmetric on $\alpha\beta$ and antisymmetric on ij determines $D^2_{aij}\Lambda$ to be

$$D^2_{aij}\Lambda = -\frac{i}{4}\gamma^b\gamma_a D_b\Lambda - \frac{i}{4}\Sigma_{[i}\Sigma^k\gamma^b\gamma_a T_{bj]k}\Lambda\ , \tag{4.1.29}$$

where we have regarded the dimension-one torsion as a matrix in spin space. In likewise fashion, the antisymmetric-symmetric part determines $D^2_{ij}\Lambda$ to be

$$D^2_{ij}\Lambda = \delta_{ij}(\frac{7}{8}B\Lambda + \frac{1}{12}\Sigma^{kl}A_{kl}\Lambda) - 2\Sigma_{(i}\Sigma^k A_{j)k}\Lambda\ , \tag{4.1.30}$$

where $A_{ij} := -\frac{1}{2}\gamma^a A_{aij}$ is regarded as a matrix in spin space. Finally, we are left with the part that is antisymmetric on both pairs of indices. After making use of the equation of motion we obtain

$$\Sigma_{ij}B\Lambda - \frac{1}{2}\Sigma^{kl}B_{ijkl}\Lambda - 4A_{ij}\Lambda - 4\Sigma_{[i}\Sigma^k A_{j]k}\Lambda + \frac{1}{3}\Sigma_{ij}\Sigma^{kl}A_{kl}\Lambda = 0\ . \tag{4.1.31}$$

This equation, cubic in Λ, has the form of an antisymmetric tensor-spinor with respect to the $SO(16)$ indices. The fact that it is identically true can be shown using the Grassmann-odd nature of Λ together with some Fierz rearrangement.

4.1.2 Dimension two

There are two Bianchi identities at dimension two, the first of which simply tells us that the Riemann tensor $R_{ab,cd}$ has the usual symmetries in the absence of torsion. In three dimensions it can be written in the form

$$R_{ab,cd} = \varepsilon_{abe}\varepsilon_{cdf}G^{ef} \tag{4.1.32}$$

where $G_{ab} := R_{ab} - \frac{1}{2}\eta_{ab}R$ is the Einstein tensor. The second Bianchi identity is

$$R_{ab,\underline{\gamma\delta}} = 2D_{[a}T_{b]\underline{\gamma\delta}} + D_{\underline{\gamma}}T_{ab\underline{\delta}} + 2T_{[a\underline{\gamma}}{}^{\underline{\epsilon}}T_{b]\underline{\epsilon\delta}} \ . \tag{4.1.33}$$

In addition, we have the dimension-two component of the Maurer-Cartan equation which gives

$$R_{abij} = 2P_a\Sigma_{ij}P_b \ . \tag{4.1.34}$$

It is a lengthy computation to analyse the content of these equations. Clearly, the $SO(16)$ curvature is immediately found from (4.1.34), while the Lorentz curvature is obtained from (4.1.33). But then there are a lot other components of (4.1.33) which must be satisfied identically. It is indeed the case that this is so, but to prove it requires further Fierz rearrangement.

For the Lorentz curvature we find

$$G_{ab} = -4(P_aP_b - \frac{1}{2}\eta_{ab}P^cP_c) + 4i(\Lambda\gamma_{(a}D_{b)}\Lambda - \eta_{ab}(B^2 - \frac{1}{3}A_{aij}A^{aij})), \tag{4.1.35}$$

where the right-hand side is essentially the on-shell energy-momentum tensor for the sigma model.

Finally, the equation of motion for the scalars can be found by acting on the fermion equation of motion with a spinorial derivative. It is

$$D^aP_a = \frac{i}{32}\Sigma^{ij}P_aA^a_{ij} - \frac{i}{16.6!}\Sigma^{i_1\ldots i_6}P_aA^a_{i_1\ldots i_6} \ . \tag{4.1.36}$$

4.2 N=8

For half-maximal supergravity in three dimensions there is a series of matter coupled geometries which make use of the cosets $(SO(8)\times SO(n))\backslash SO(8,n)$

[66]. Notice this implies that the local R-symmetry group will be enlarged by the $SO(n)$ factor and hence that there will be a corresponding additional curvature tensor in the geometry. We shall denote $SO(8,n)$ vector indices by $R, S, \ldots$ and $SO(n)$ vector indices by $r, s \ldots$, so $R = (i, r)$ where i is an $SO(8)$ vector index as before. We take the generators of $\mathfrak{so}(8,n)$, $M_{RS} = -M_{SR}$, to satisfy

$$[M_{RS}, M^{TU}] = -4\delta_{[R}{}^{[T} M_{S]}{}^{U]} \tag{4.2.1}$$

Written out with respect to the $\mathfrak{so}(8) \oplus \mathfrak{so}(n)$ decomposition this is

$$\begin{aligned}
[M_{ij}, M^{kl}] &= -4\delta_{[i}{}^{[k} M_{j]}{}^{l]} \\
[M_{rs}, M^{tu}] &= -4\delta_{[r}{}^{[t} M_{s]}{}^{u]} \\
[M_{ij}, M^{kr}] &= -2\delta_{[i}{}^{k} M_{j]}{}^{r} \\
[M_{rs}, M^{kt}] &= 2\delta_{[r}{}^{t} M_{s]}{}^{k} \\
[M_{ir}, M_{js}] &= -\eta_{ij} M_{rs} - \eta_{rs} M_{ij} \ .
\end{aligned} \tag{4.2.2}$$

The $SO(8,n)$ metric is $\eta_{RS} = (\delta_{ij}, -\delta_{rs})$.

The sigma model field $\mathcal{V}$ is an element of $SO(8,n)$ that depends on the superspace coordinates. It is acted on to the right by $SO(8,n)$ and to the left by the local $SO(8) \times SO(n)$ and therefore corresponds to an $(SO(8) \times SO(n))\backslash SO(8,n)$ sigma model superfield. The Maurer-Cartan form is

$$\Phi := d\mathcal{V}\mathcal{V}^{-1} := P + Q \ , \tag{4.2.3}$$

where $Q = \frac{1}{2}\Omega^{ij} M_{ij} + \frac{1}{2}\Omega^{rs} M_{rs}$, Ω^{rs} being the $\mathfrak{so}(n)$ connection and where $P = P^{ir} M_{ir}$ takes its values in the quotient algebra. From the Maurer-Cartan equation (vanishing $SO(8,n)$ curvature), $d\Phi + \Phi^2 = 0$, we find

$$
\begin{aligned}
DP &= 0 &\quad (4.2.4)\\
R &= -P^2 \ , &\quad (4.2.5)
\end{aligned}
$$

where $R := \frac{1}{2}R^{ij}M_{ij} + \frac{1}{2}R^{rs}M_{rs}$ is the $\mathfrak{so}(8) \oplus \mathfrak{so}(n)$ curvature, while D is the corresponding covariant exterior derivative. In indices, the above equations are

$$
\begin{aligned}
2D_{[A}P_{B]} + T_{AB}{}^{C}P_C &= 0 &\quad (4.2.6)\\
R_{AB} &= [P_A, P_B] \ . &\quad (4.2.7)
\end{aligned}
$$

The dimension of the sigma model coset is $8n$, so we need an equal number of fermions for supersymmetry. To ensure this we impose a constraint on the dimension one-half component of P. We set

$$
P^{ir}_{\alpha I} = i(\Sigma^i)_{IJ'}\Lambda^r_{\alpha J'} \ , \qquad (4.2.8)
$$

where $\Lambda^r_{\alpha J'}$ describes the $8n$ physical one-half fields. The dimension-one component of (4.2.7) is then satisfied if

$$
D_{\alpha I}\Lambda^r_{\beta J'} = \frac{1}{2}(\gamma^a)_{\alpha\beta}(\Sigma_i)_{IJ'}P^{ir}_a \ . \qquad (4.2.9)
$$

We can think of P_{aI} as essentially the spacetime derivative of the physical scalar fields. In order to see this more explicitly, it is perhaps useful to look at the linearised limit. In the physical gauge we can put $\mathcal{V} = \exp(\phi^{ir}M_{ir})$ where ϕ^{ir} denotes the $8n$ scalars. If we now keep only terms linear in the fields we find

$$
\begin{aligned}
D_{\alpha I}\phi^{ir} &= i(\Sigma^i \Lambda^r_\alpha)_I \\
D_{\alpha I}\Lambda^r_{\beta J'} &= \frac{1}{2}(\gamma^a)_{\alpha\beta}(\Sigma_i)_{IJ'}P^{ir}_a = \frac{1}{2}(\gamma^a)_{\alpha\beta}(\Sigma_i)_{IJ'}\partial_a\phi^{ir} \ , \qquad (4.2.10)
\end{aligned}
$$

where $D_{\alpha I}$ here is now the usual supercovariant derivative in flat superspace. It follows from (4.2.10) that, in the linearised limit, both ϕ^{ir} and $\Lambda^r_{\alpha I'}$ satisfy free field equations of motion. To see this explicitly one needs to apply another spinorial derivative to the second of these equations and use the supersymmetry algebra to find the Dirac equation. The scalar equation then follows from this by applying another derivative.

It is now easy to compute the dimension-one curvature and torsion in terms of the sigma model fields and to verify that they can be slotted into the superconformal geometry. We find

$$
\begin{aligned}
K_{IJ} &= \frac{i}{4}\delta_{IJ}B := \frac{i}{4}\delta_{IJ}\Lambda\Lambda \\
L_{aIJ} &= \frac{i}{8}(\Sigma^{ij})_{IJ}A_{aij} := \frac{i}{8}(\Sigma^{ij})_{IJ}\Lambda\gamma_a\Sigma_{ij}\Lambda \\
M_{IJKL} &= \frac{i}{32}(\Sigma^{ij})_{IJ}(\Sigma^{kl})_{KL}B_{ijkl} \\
&:= \frac{i}{32}(\Sigma^{ij})_{IJ}(\Sigma^{kl})_{KL}\Lambda\Sigma_{ijkl}\Lambda \ , \qquad (4.2.11)
\end{aligned}
$$

where, on the right-hand-side, the spacetime and internal spinor indices are contracted in the natural way (see appendix). The internal $SO(n)$ vector indices are contracted with η_{rs}. These formulae determine the non-zero dimension-one torsion and curvature components. The dimension-one component of the $SO(n)$ curvature is

$$R_{\alpha I\beta J,rs} = 2(\gamma^a)_{\alpha\beta}\delta_{IJ}\Lambda_r\gamma_a\Lambda_s - \frac{1}{2}\varepsilon_{\alpha\beta}(\Sigma^{ij})_{IJ}\Lambda_r\Sigma_{ij}\Lambda_s \ . \tag{4.2.12}$$

Notice that M_{IJKL} is in the representation (0002) (because Λ carries a primed spinor index), so that it is anti-self-dual.

Equations (4.2.11) and (4.2.12) show that the geometry is determined in terms of the matter fields, and so the full non-linear equations of motion for the physical fields can be derived from the above set of equations by supersymmetry.

4.2.1 Vector fields

Later on we will discuss the gauged geometry of the above theories and for this we need the two-form field strengths. These should transform according to a representation of the duality group $SO(8, n)$ and by Hodge duality there should be the same number of them as there are scalars. This is accomplished by taking the vector fields to transform under the adjoint representation of $SO(8, n)$. It will turn out that $8n$ of the field strengths are essentially duals of the field strengths for the scalars at dimension one while the others are composite. In the ungauged theory the Bianchi identities for the two-forms are abelian,

$$dF^{RS} = 0 \ . \tag{4.2.13}$$

It is not difficult to solve for the components of F^{RS} in terms of the physical fields. We denote the components of the scalar field matrix $\mathcal{V}$ in the fundamental representation by

$$\mathcal{V}_{\bar{R}}{}^R = (V_i{}^R, V_r{}^R) \ . \tag{4.2.14}$$

Then the components of F^{RS} are

$$\begin{aligned}
F^{RS}_{\alpha I\beta J} &= i\varepsilon_{\alpha\beta}(\Sigma^{ij})_{IJ}V_i{}^R V_j{}^S \\
F^{RS}_{a\beta J} &= -2i(\gamma_a\Sigma^i\Lambda^r)_{\beta J}V_r{}^{[R}V_i{}^{S]} \\
F^{RS}_{ab} &= \varepsilon_{ab}{}^c(2P_c^{ir}V_r{}^{[R}V_i{}^{S]} + \frac{3i}{4}A_a^{ij}V_i{}^R V_j{}^S - 2iA_a^{rs}V_r{}^R V_s{}^S)\ .
\end{aligned} \tag{4.2.15}$$

The bilinear (in Λ) A_{aij} is defined in (4.2.11) above, while $A_{ars} := \Lambda_r\gamma_a\Lambda_s$. Notice that this equation shows that the dimension-one component of F^{RS} contains the $8n$ scalar field strengths P_a^{ir} as required.

4.3 IIA and IIB

In the next chapter we will discuss the form fields of type IIA and IIB supergravity. We therefore briefly review these theories and also give some references here.

The complete IIB supergravity (for the physical fields) was written down in superspace in [111], the component version having been given in [112]. The dual forms were added in [113, 114], and all of the forms up to degree eleven in [74]. The conventions we follow here are those of [74] although we have slightly changed the normalisations of some of the forms and written them with upper $SL(2,\mathbb{R})$ indices. One can transform from these to those of [74] by means of the ε-tensor. In the original paper a complex $U(1)$ notation was used for the spinors, but it is probably more convenient to use a real $SO(2)$ notation, as in [115], where the relation between the conventions of [111] and [74] can be found.

The dimension-zero torsion is given by

$$T_{\alpha i\beta j}{}^{c} = i\delta_{ij}(\gamma^{c})_{\alpha\beta}\ , \tag{4.3.1}$$

where $i, j = 1, 2$ are $SO(2)$ spinor indices (we use $r, s, \ldots$ for vector indices). The geometric tensors cannot contain the scalar fields, as the formalism is $SL(2,\mathbb{R})$ covariant, as well as having a local $U(1)$ symmetry for which the gauge field is a composite constructed from the scalars in the usual manner. The spin one-half fields are found in the dimension-one-half torsion, while the other physical field strengths arise at higher dimension, although the (bosonic) five-form does not appear directly.

In the text we gave the forms in an $SL(2,\mathbb{R})$ basis, but it is sometimes convenient to use the $SO(2)$ basis, the two being related by the scalar matrix $\mathcal{V}_r{}^R$. In this basis the Bianchi identities take the form

$$DF = FF - F \wedge P \tag{4.3.2}$$

where D is covariant with respect to $SO(2)$, FF denotes the same term that occurs in the $SL(2,\mathbb{R})$ basis, except that the indices are now lower case, and P denotes the matrix of one-forms in the representation appropriate to the form F on the left. One advantage of this basis is that the scalars cannot appear undifferentiated so that the dimension-zero components are simply given by products of (16×16) gamma-matrices and $SO(2)$ gamma-matrices, $(\tau^r)_{ij}$ (symmetric, traceless), δ_{ij} or ε_{ij}. We refer the reader to [74] for details.

The IIA theory was written down in components in [116] and in superspace in [117]. It was also derived by superspace dimensional reduction from $D = 11$ in [83].[2] The version we use here was briefly outlined in [75]. We use thirty-two component Majorana spinors. The dimension-zero torsion is

$$T_{\alpha\beta}{}^{c} = -i(\Gamma^{c})_{\alpha\beta}\ . \tag{4.3.3}$$

[2] In appendix C of [83] equation (C.7) should read $\chi_\alpha = i(\Gamma_{11}\nabla)_\alpha\Phi$.

The string frame is used, so that the dimension-zero component of H_3 has no factor of the dilaton, $H_{1,2} \propto \tilde{\Gamma}_{1,2}$, while the dimension-zero components of the RR forms all have a factor of $e^{-\phi}$, multiplied by appropriate gamma-matrices. Since $dG_{2n+2} = H_3 G_{2n}$, this implies that the dimension-zero components of the RR forms have a factor of Γ_{11} for n even, but not for n odd.

Chapter 5

Form fields

In recent years there have been several studies of the systematics of form fields in supergravity theories, starting with [67, 68]. It was realised that this could be formalised in terms of Borcherds algebras [39], and also in terms of E_{11} [69, 70, 71, 43]. (See [72] for a discussion of the relation between the two). In a separate, but related development, it has been shown that the same sets of forms contribute to the hierarchies found in gauged supergravity theories [73]. A key feature is that these forms fall into representations of the duality groups. In addition to the physical forms and their duals there are also $(D-1)$-form potentials, related to gaugings, and D-form potentials, related to space-filling branes. We discuss these fields for all theories discussed in the previous chapter and show that all of the coupled Bianchi identities for the associated field strengths are satisfied.

Our construction of the allowed form fields uses only supersymmetry and duality symmetry. The superspace method has some advantages, especially for the D-form potentials. This is because it makes sense to consider $D+1$-(and indeed higher)-form field strengths in superspace due to the fact that the odd basis differential forms are commutative. This point of view was advocated previously in the context of maximal supersymmetry in ten dimensions

[74, 75]. An additional feature of the formalism is that it is manifestly supersymmetric, and indeed, if one concentrates on the field strengths, manifestly covariant under all symmetries.

The method of constructing the allowed form fields is straightforward. One starts off with a set of physical forms, including the duals, and then asks how many further forms can be constructed that satisfy consistent Bianchi identities, of the type $dF = F^2$, and that also transform under appropriate representations of the duality group, when present. Here, consistency just means that applying a second d must give zero. When one does this one will obtain an expression cubic in the Fs on the right which must vanish. The form fields that are allowed are simply those satisfying these conditions.

In general, field strength forms with degree greater than $(D + 1)$ vanish in supergravity, but this does not mean that these forms are not of interest. There are some examples of non-vanishing $(D + 2)$ forms, including in IIA supergravity in $D = 10$, while other forms may have interesting Bianchi identities of the form $dF = F^2$ where the two Fs on the right-hand side do not vanish even though the left side does identically. An example of this occurs in the IIB theory where there are thirteen-forms whose Bianchi identities involve non-zero lower-degree forms on the right. The $D + 2$-forms can also play a rôle in the gauge hierarchy [73].

More importantly, perhaps, is that even though the form fields with degree greater that $D + 2$ vansih in the supergravity limit this need not be so when string corrections are taken into account. Later on we will be interested in whether any of them could indeed become non-zero. This possibility will be investigated in the final chapter by looking at a subset of the possible six-forms that can arise and give some evidence that one can indeed find some non-vanishing six-form components that are compatible with at least some of the Bianchi identities. This result gives us confidence that forms beyond the spacetime limit are indeed physically significant when one takes higher-order corrections into account.

In the following chapter we derive the form spectrum for $D = 3$ half-maximal and maximal Poincaré supergravity and also for type IIA and IIB supergravity. We have chosen these seemingly arbitrary theories are for their different properties. Type IIA and IIB supergravity have very simple duality groups (IIA even lacks a duality group), which clarifies the algebraic structure. Half-maximal supergravity in three dimensions is instead suitable for studying the rôle of the higher-degree forms in the presence of string corrections. In the case of the ten-dimensional theories, we expect these to begin at order α'^3. On dimensional grounds, this means that the first form which is zero in supergravity but not necessarily in string theory, is the $(0, 13)$-components of the thirteen-forms. To investigate whether such a Bianchi identity is satisfied involves the tensor product of thirteen 32-component spinors. In the half-maximal theory in three dimensions we instead expect string corrections to start at order α' so that already the six-forms can become non-zero. Finally, in the case of maximal supergravity in three dimensions, we have set out to describe the complete geometry, hence in this case we study the form fields for their own sake.

5.1 Consistent Bianchi identities

To determine if a form field transforming under some representation $\mathcal{R}_\ell$ of the duality group is allowed by supersymmetry one needs to construct its Bianchi identity, confirm that it is consistent and also that there are no obstructions to it being satisfied. The representations $\mathcal{R}_\ell$ are contained in the product of the representations that the lower-degree forms transform under.

The Bianchi identity for a $\ell + 1$-form field strength transforming under the representation $\mathcal{R}_\ell$ is the $\ell + 2$-form

$$I_{\ell+2}{}^{\mathcal{R}_\ell} = dF_{\ell+1}{}^{\mathcal{R}_\ell} - \sum_{m+n=\ell} F_{m+1}{}^{\mathcal{R}_m} \wedge F_{n+1}{}^{\mathcal{R}_n} a_{\mathcal{R}_m \mathcal{R}_n}{}^{\mathcal{R}_\ell}, \tag{5.1.1}$$

where the sum on the right hand side is over all field strengths for which $\mathcal{R}_\ell$ is contained in the direct product $\mathcal{R}_m \otimes \mathcal{R}_n$ [1]. The consistency of a Bianchi identity means that taking the exterior derivative of (5.1.1) gives zero,

$$\begin{aligned} dI_{\ell+2}{}^{\mathcal{R}_\ell} = 0 = & \sum_{m+n=\ell} F_{m+1}{}^{\mathcal{R}_m} \wedge \sum_{r+s=n} \left(F_{r+1}{}^{\mathcal{R}_r} \wedge F_{s+1}{}^{\mathcal{R}_s} b_{\mathcal{R}_r\mathcal{R}_s}{}^{\mathcal{R}_n}\right) a_{R_m R_n}{}^{R_\ell} \\ + \quad (-1)^{2m+n} & \sum_{m+n=\ell} F_{n+1}{}^{\mathcal{R}_n} \wedge \sum_{t+u=m} \left(F_{t+1}{}^{\mathcal{R}_t} \wedge F_{u+1}{}^{\mathcal{R}_u} c_{\mathcal{R}_t\mathcal{R}_u}{}^{\mathcal{R}_m}\right) a_{\mathcal{R}_m\mathcal{R}_n}{}^{\mathcal{R}_\ell}, \end{aligned} \tag{5.1.2}$$

where again terms on the right-hand-side of (3.2.11) is non-zero whenever $\mathcal{R}_\ell$ lies in the product $\mathcal{R}_r \otimes \mathcal{R}_s \otimes \mathcal{R}_n$ or $\mathcal{R}_t \otimes \mathcal{R}_u \otimes \mathcal{R}_m$. The degeneracy of a form field transforming under a certain representation is the number of ways one can make (5.1.2) consistent.

When a Bianchi identity is consistent one needs to show that it is satisfied, for lower-degree forms this needs to be done explicitly, at least for its lowest dimensional components. For higher-degree forms, the use of superspace cohomology guarantees they are satisfied provided they are consistent. To show this we need to give a short introduction to superspace cohomology.

5.2 Superspace cohomology

Since the tangent bundle in superspace splits into even and odd parts it is possible to split the space of n-forms into spaces of (p,q)-forms, $p+q=n$, where a (p,q) form has p even and q odd indices:

---

[1] $\mathcal{R}_\ell$, $\mathcal{R}_m$... are one of the representations of the duality group forms of degree $\ell+1$, $m+1$... transform under. a, b... are made up of invariants of the duality group projecting $\mathcal{R}_m \otimes \mathcal{R}_n$ onto $\mathcal{R}_\ell$...

$$\Omega^{p,q} \ni \omega_{p,q} = \frac{1}{p!q!} E^{\beta_q} \ldots E^{\beta_1} E^{a_p} \ldots E^{a_1} \omega_{a_1 \ldots a_p \beta_1 \ldots \beta_q} \ . \tag{5.2.1}$$

The exterior derivative splits into four terms with different bidegrees:

$$d = d_0 + d_1 + t_0 + t_1 \ , \tag{5.2.2}$$

where the bidegrees are $(1,0),(0,1),(-1,2)$ and $(2,-1)$ respectively. The first two, d_0 and d_1, are essentially even and odd differential operators, while the other two are algebraic operators formed with the dimension-zero and dimension three-halves torsion respectively. In particular,

$$(t_0 \omega_{p,q})_{a_2 \ldots a_p \beta_1 \ldots \beta_q} \propto T_{(\beta_1 \beta_2}{}^{a_1} \omega_{a_1 | a_2 \ldots a_p | \beta_3 \ldots \beta_{q+2})} \ . \tag{5.2.3}$$

The equation $d^2 = 0$ splits into various parts according to their bidegrees amongst which one has

$$\begin{aligned} (t_0)^2 &= 0 && (5.2.4)\\ t_0 d_1 + d_1 t_0 &= 0 && (5.2.5)\\ d_1^2 + t_0 d_0 + d_0 t_0 &= 0 \ . && (5.2.6) \end{aligned}$$

The first of these enables us the define the cohomology groups $H_t^{p,q}$, the space of t_0-closed (p,q)-forms modulo the exact ones [91]. The other two then allow one to define the spinorial cohomology groups $H_s^{p,q}$, but we shall not need these here.

5.3 Soluble Bianchi identities

Suppose that we have a closed n-form whose lowest component (i.e. the one with highest odd degree) is $\omega_{p,q}$. This component will have to be t_0-closed, and hence, unless it is exact, will be determined by an element of the cohomology group $H_t^{p,q}$. The key point is that, in $N = 2, D = 10$ supersymmetry, all of these groups vanish for $p > 1$, and for the three-dimensional supergravity theories, $H_t^{p,q}$ vanish for $p \geq 1$. As we shall see this makes the analysis of the consistency of the Bianchi identities almost trivial.

Let us write the Bianchi identities from above in a simpler form

$$I_{\ell+1}{}^{\mathcal{R}_\ell} = dF_n{}^{\mathcal{R}_\ell} - (FF)_{\ell+1}{}^{\mathcal{R}_\ell} \ , \qquad (5.3.1)$$

where $\mathcal{R}_\ell$ again denotes the representation of the duality group G under which F_ℓ transforms. Even if a particular Bianchi is not satisfied, consistency means that $dI_{n+1}{}^{\mathcal{R}_\ell} = 0$. Since there are no fields in supergravity that have negative dimensions, the lowest non-vanishing component of any I also has dimension zero and is given by $I_{n-3,4}$. This must be t_0-closed and will therefore be t_0-exact if $\ell \geq 5$ in type IIA and IIB supergravity, and if $\ell \geq 4$ in the three-dimensional supergravity theories. Now we know that the Bianchi identities are satisfied for all of the physical fields, so we can deduce from this that for all of the other forms, the physical duals and the D forms and $D+1$ as well as any higher degree forms, the lowest components of the corresponding Bianchi identities are t_0-exact.[2] Thus we will have $I^X_{n-3,4} = t_0 J^X_{n-2,2}$ for some J. But $J^X_{n-2,2}$ has precisely the same index structure as the lowest non-zero component of F_n, namely $F_{n-2,2}{}^{\mathcal{R}_\ell}$, and hence setting $J_{n-2,2}{}^{\mathcal{R}_\ell} = 0$ allows one to solve for it without imposing any further constraints. We can therefore do this and turn our attention to the next level, $I_{n-2,3}{}^{\mathcal{R}_\ell}$, which is also t_0-exact and which has the right number of components to allow us to solve for

[2] This has to be be done sequentially in order of increasing form degree.

${F_{n-1,1}}^{\mathcal{R}_\ell}$. Going one step further in a similar fashion we see that we will be able to solve for ${F_{n,0}}^{\mathcal{R}_\ell}$ and hence for the whole of ${F_n}^{\mathcal{R}_\ell}$.

To summarise, when the Bianchi identities for the physical fields are satisfied, there is no obstruction to their being solved for the higher-rank form fields provided that the Bianchi identities themselves are formally consistent.

5.4 IIB

5.4.1 Up to spacetime limit

The bosonic spectrum of the IIB theory consists of the graviton, two scalar fields, the dilaton and axion, a pair of two-form potentials and a four-form potential whose five-form field strength is self-dual. To these we can add their duals, a doublet of seven-form field strengths and a triplet of nine-forms. The latter are dual to the field strengths for the scalars and transform under the triplet representation of $SL(2,\mathbb{R})$ even though there are only two scalars. This can be achieved by means of a constraint on the field strength that ensures that there are only two dynamical dual eight-form potentials. This set can then be extended by a quadruplet and a doublet of eleven-forms, corresponding to the ten-form potentials first introduced in [105]. The set of forms is then $\{F_3^R, F_5, F_7^R, F_9^{RS}, F_{11}^{RST}, F_{11}^R\}$, together with the one-form field strengths for the scalars. The Bianchi identities for these forms are:

$$\begin{aligned}
dF_3^R &= 0 \\
dF_5 &= \varepsilon_{RS} F_3^R F_3^S \\
dF_7^R &= F_3^R F_5 \\
dF_9^{RS} &= F_3^{(R} F_7^{S)} \\
dF_{11}^{RST} &= F_3^{(R} F_9^{ST)} \\
dF_{11}^R &= \varepsilon_{ST} F_3^S F_9^{TR} + \frac{3}{4} F_5 F_7^R \ .
\end{aligned} \tag{5.4.1}$$

The scalars can be described by an element ${\mathcal{V}_r}^R$ of $SL(2,\mathbb{R})$ modulo local $U(1)$ gauge transformations, where r is a local $SO(2)$ vector index. The Maurer-Cartan form $d\mathcal{V}\mathcal{V}^{-1} = P + Q$, where Q is the $U(1)$ connection and P can be considered as the one-form field strength for the scalars. It carries local $SO(2)$ indices and satisfies $DP = 0$, but we can convert these indices to global ones by multiplying by two factors of $\mathcal{V}$ to form the $SL(2,\mathbb{R})$ triplet of one-forms $F_1^{RS} := \delta^{rt} {P_t}^s {\mathcal{V}_r}^R {\mathcal{V}_s}^S$. The Bianchi identity for F_1^{RS} is simply $dF_1^{RS} = 0$, and indeed one can solve it by setting $F_1^{RS} = \frac{1}{2} dM^{RS}$, where $M^{RS} := \delta^{rs} {\mathcal{V}_r}^R {\mathcal{V}_s}^S$.

It is a simple matter to check that the Bianchi identities (5.4.1) are indeed consistent. We can show that there are no further gauge-trivial identities ($dF = 0$), except with degree eleven, using an argument similar to that in the previous section. In IIB a gauge-trivial identity could in principle be any one for the duals or non-physical forms, if it were to turn out that the right-hand sides of any of equations (5.4.1) could be set to zero. We can see this by applying a similar argument to the one used for the Bianchi identities which implies that the lowest component of such an F, $F_{n-2,2}$, has to be exact, $F_{n-2,2} = t_0 G_{n-1,0}$, since $(n-2) > 2$. Iterating this one finds that F itself must be exact, $F_n = dG_{n-1}$, say. But the only non-zero component of G_{n-1} is $G_{n-1,0}$ which has to have dimension zero. This can only be some

Lorentz-invariant tensor times a function of the scalars, and could therefore only be $\varepsilon_{a_1\dots a_{10}}$, or $\varepsilon_{10,0}$ in form notation. Thus there is an exact gauge-trivial eleven-form that is a singlet under the duality group.

5.4.2 Beyond the spacetime limit

We start by considering the thirteen-forms in IIB supergravity. There are three possibilities corresponding to the five-, three- and one-dimensional representations of $SL(2,\mathbb{R})$:

$$\begin{aligned} dF_{13}^{RSTU} &= F_3^{(R}F_{11}^{STU)} \\ dF_{13}^{RS} &= \varepsilon_{UV}F_3^U F_{11}^{VRS} + \frac{8}{15}F_3^{(R}F_{11}^{S)} + \frac{2}{5}F_5F_9^{RS} \\ dF_{13} &= \varepsilon_{RS}F_3^R F_{11}^S + \frac{3}{8}\varepsilon_{RS}F_7^R F_7^S \ . \end{aligned} \tag{5.4.2}$$

Using (5.4.1) one can easily see that the first of these is consistent. The other two require a bit more work but turn out to be consistent for the given choice of constants. Any thirteen-form must be zero in supergravity since its dimension-zero component is $F_{11,2}$. However, all of the forms appearing on the right-hand side of the above equations are non-zero. Moreover, the dimension-zero component of any of the Bianchi identities has the form $I_{10,4}$ and so need not vanish identically. Nevertheless, since the Bianchi identities are consistent, it follows that $t_0 I_{10,0} = 0$ and this implies that $I_{10,4} = 0$, so that these equations are guaranteed to be satisfied.

Moving on to the fifteen-forms, we find the following set of possibilities

$$
\begin{aligned}
dF_{15}^{RSTUV} &= F_3^{(R}F_{13}^{STUV)} && (5.4.3)\\
dF_{15}^{RST} &= a\varepsilon_{UV}F_3^U F_{13}^{VRST} + bF_3^{(R}F_{13}^{ST)} + cF_5F_{11}^{RST}\\
&\quad +dF_7^{(R}F_9^{ST)} && (5.4.4)\\
dF_{15}^{R} &= e\varepsilon_{ST}F_3^S F_{13}^{TR} + fF_3^R F_{13} + gF_5F_{11}^R + h\varepsilon_{ST}F_7^S F_9^{TR}\ . && (5.4.5)
\end{aligned}
$$

Applying d to the second of these we find two constraints on the constants a, b, c, d coming from terms with $\varepsilon_{UV}F_3^U F_3^V F_{11}^{RST}$ and $F_5F_3^{(R}F_9^{ST)}$ so that we can eliminate two of them, say c and d. We therefore find that there are two independent fifteen-forms in this representation whose Bianchi identities can be combined into

$$
\begin{aligned}
dF_{15}^{RST} &= a(\varepsilon_{UV}F_3^U F_{13}^{VRST} + \frac{5}{8}F_5F_{11}^{RST} - \frac{5}{8}F_7^{(R}F_9^{ST)}) +\\
&\quad +b(F_3^{(R}F_{13}^{ST)} - \frac{1}{2}F_5F_{11}^{RST} + \frac{9}{10}F_7^{(R}F_9^{ST)})\ . && (5.4.6)
\end{aligned}
$$

For the doublet representation (5.4.5) we find three possible consistency conditions from terms with $\varepsilon_{ST}F_3^S F_3^T F_{11}^R$, $\varepsilon_{ST}F_3^S F_5 F_9^{TR}$ and $F_3^R\varepsilon_{ST}F_7^S F_7^T$. However, only two of them are independent and we therefore have two fifteen-forms in the doublet representation. Their Bianchi identities can be written

$$
\begin{aligned}
dF_{15}^{R} &= e(\varepsilon_{ST}F_3^S F_{13}^{TR} + \frac{2}{5}F_5F_{11}^R) +\\
&\quad +c(F_3^{(R}F_{13} - \frac{1}{2}F_5F_{11}^R + \frac{1}{2}\varepsilon_{ST}F_7^S F_9^{TR})\ . && (5.4.7)
\end{aligned}
$$

The fifteen-forms vanish identically in supergravity, but not all of the forms on the right-hand side are zero. However, in this case the dimension-zero component of a Bianchi identity has the form $I_{12,4}$ and so must vanish by antisymmetry.

This analysis shows that at each level the number of representations that can arise increases, and that, from degree fifteen onwards, there are also multiplicities in some of the representations.

5.5 IIA

5.5.1 Up to spacetime limit

The situation in IIA is similar, but there are two differences: there is no duality group and the forms can have both even and odd degree. The physical forms are the RR two- and four-forms, and the NS three-forms; their duals are RR six- and eight-forms and an NS seven-form, together with a nine-form which is dual to the one-form field strength of the dilaton. The RR Bianchi identities, including one for the ten-form, are

$$dG_{2n+2} = H_3 G_{2n} \qquad \text{for } n = 0, 1, 2, 3, 4 \ , \tag{5.5.1}$$

where G_0 is taken to be zero. (We do not consider a mass deformation.) The Bianchi identities for the NS forms up to degree nine are

$$\begin{aligned} dH_3 &= 0\,, \\ dH_7 &= \frac{1}{2}G_4^2 - G_2 G_6\,, \\ dH_9 &= -H_3 H_7 + \frac{1}{2}G_4 G_6 - \frac{3}{2}G_2 G_8 \ . \end{aligned} \tag{5.5.2}$$

Now consider the possible eleven-form field strengths. There are two allow-able Bianchi identities that can be combined into one:

$$\begin{aligned} dH_{11} &= A(H_3H_9 + \frac{3}{2}G_2G_{10} - \frac{1}{4}G_6^2) \\ &\quad +B(-G_2G_{10} + G_4G_8 - \frac{1}{2}G_6^2) \quad , \end{aligned} \tag{5.5.3}$$

where A and B are real constants.

All of these Bianchi identities are consistent [75], and so we know that they will all be satisfied given that the physical ones are (i.e. those for the two-, three- and four-forms). In IIA $H_t^{p,q} = 0$ if $p > 1$, and if $p = 1$ the basic non-trivial element is $\tilde{\Gamma}_{1,2} \in H_t^{1,2}$, where $\Gamma_{p,2}$ denotes a symmetric gamma-matrix with p spacetime indices and the tilde indicates the presence of a factor of Γ_{11}. One possibility for the eleven-forms is that both A and B are zero. In this case we have a gauge-trivial eleven-form, but by the same cohomological argument that we used in IIB, it is exact.

It is easy to verify that there are two non-trivial eleven-forms by looking at the dimension-zero components. For

$$H_{9,2} = -iKe^{-2\phi}\Gamma_{9,2} \quad , \tag{5.5.4}$$

with K constant, we find that (5.5.3) is satisfied if $2A + 8B = K$, so that there are indeed two independent gauge non-trivial eleven-forms. The other possibility, namely $H_{9,2} \sim i\tilde{\Gamma}_{9,2}$ corresponds to the gauge-trivial case and so requires $A = B = 0$. It is exact, i.e. proportional to $t_0\varepsilon_{10,0}$.

5.5.2 Beyond the spacetime limit

Firstly we observe that the Bianchi identities (5.5.1) for the RR forms are consistent for any value of n. For the most part these are trivial in supergravity, but there is a non-zero RR twelve-form with dimension-zero component

$$G_{10,2} = -iKe^{-\phi}\Gamma_{10,2} \tag{5.5.5}$$

for some real non-zero constant K. In fact the Bianchi identity $dG_{12} = H_3G_{10}$ is automatically soluble, by cohomology, the solution being given by the above expression. (The higher-dimensional components of G_{12} are identically zero.)

One can also have non-trivial thirteen-form Bianchi identities in IIA. The consistent ones turn out to be

$$\begin{aligned} dH_{13} \;=\;& A(-\frac{5}{4}G_2G_{12} - \frac{1}{4}G_4G_{10} + \frac{1}{4}G_6G_8) \\ &+B(\frac{5}{2}G_2G_{12} - \frac{3}{2}G_4G_{10} + \frac{1}{2}G_6G_8) + H_3H_{11}\;. \end{aligned} \tag{5.5.6}$$

where A and B are the two constants that appear in (5.5.3). Since there are two H_{11}s depending on the choice of these, there are also two independent thirteen-form Bianchi identities.

As in the IIB case, the left-hand side of this equation is identically zero in supergravity, and the fact that the right-hand side vanishes as well, even though the individual forms that appear there do not, follows from cohomology.

5.6 N=16

In this section we derive the set of forms for maximal supergravity in three dimensions. We start of with the physical one-forms P. In addition to these we are allowed to introduce their dual two-forms which transform under the 248 of E_8. Beyond these, we will have three- and four-forms in arbitrary

E_8 representations as the bosonic potentials do not introduce any new independent degrees of freedom. We also calculate the possible five-forms, their lowest dimensional components need not be zero and they are in fact needed in the complete gauged theory. The Bianchi identities for the two-forms is

$$dF_2^R = 0 \ , \tag{5.6.1}$$

where $R, S, T = 1, \ldots 248$ transforming under the adjoint representation of $E8$. We shall see shortly that this identity is indeed satisfied, and that the $(2,0)$ component of F_2 is given by the dual of P_a together with a bilinear fermion contribution. The components of the forms in an E_8 basis generically involve the scalar field matrix $\mathcal{V}$ which we write, in the adjoint representation as,

$$\mathcal{V}_{\bar{R}}{}^R = \left(V_{ij}{}^R, V_I{}^R\right) \ , \tag{5.6.2}$$

where the barred index is to be acted on by $SO(16)$ and therefore splits into the appropriate representations determined by the branching rules. The dimension-zero component of F_2^R is given by

$$F^R_{\alpha i \beta j} = -2i\varepsilon_{\alpha\beta} V_{ij}{}^R \ . \tag{5.6.3}$$

Now suppose we have a set of three-forms $F_3^{\mathcal{X}}$ transforming under some representation labelled by $\mathcal{X}$. The Bianchi identity has the form

$$dF_3^{\mathcal{X}} = F_2^S F_2^R t_{RS}{}^{\mathcal{X}} \ , \tag{5.6.4}$$

where $t_{RS}{}^{\mathcal{X}}$ is an E_8-invariant tensor.

The symmetric product of two 248s is $1 + 3875 + 27000$. For the singlet the Bianchi identity is

$$dF_3 = F_2^R F_2^S a_{RS} \ , \tag{5.6.5}$$

where a_{RS} is the E_8 metric (given in appendix A.2). The dimension-zero component is

$$F_{a\beta j\gamma k} = -i\delta_{jk}(\gamma_a)_{\beta\gamma} \ . \tag{5.6.6}$$

It is not difficult to see that $I_{0,4} = 0$, and so we conclude that there is a singlet three-form.

Next consider the 3875. The branching rule is $3875 \rightarrow 135{+}1820{+}1920'$. The 135 is a symmetric traceless tensor which we shall write as $t_{i,j}$, the 1820 is a fourth-rank antisymmetric tensor, and the $1920'$ is a sigma-traceless primed vector-spinor. The dimension-zero component of this three-form is

$$F^U_{a\beta j\gamma k} = -i(\gamma_a)_{\beta\gamma} V_{j,k}{}^U \ , \tag{5.6.7}$$

where $U, V, W = 1, \ldots 3875$ and where $\mathcal{V}_{\bar{U}}{}^U$ is the scalar field matrix in the 3875 representation, so that $V_{i,j}{}^U$ is the projection onto the 135 in $\bar{U}$. If we write the Bianchi identity as $I_4^U = dF_3^U - F_2^S F_2^R b_{RS}{}^U = 0$, we can see that its dimension-zero component is indeed satisfied. This is because the symmetrised product of two 120s coming from $F^S_{0,2} F^R_{0,2} b_{RS}{}^U$ can give both 135 and 1820 (since these are both contained in 3875), but the 1820 drops out in the Bianchi identity because the symmetrisation over the four odd indices would require antisymmetrisation over the four two-component Lorentz spinor indices. We can thus conclude without any further calculation that this Bianchi identity is satisfied.

The final possibility for three-forms is the 27000. The branching rule is $27000 \rightarrow 1 + 1820 + 6435 + 5304 + 128 + 13312$, the last two being spinorial representations. The most significant one for us is the 5304; this is a tensor with the symmetries of the Weyl tensor (in sixteen dimensions). The only possibility for the dimension-zero component is

$$F^{\mathcal{X}}_{a\beta j\gamma k} = -i\delta_{jk}(\gamma_a)_{\beta\gamma}V_0{}^{\mathcal{X}} \ , \tag{5.6.8}$$

where $V_0{}^{\mathcal{X}}$ denotes the singlet projection of the scalar matrix in the 27000. However, the dimension-zero $(F_2)^2$ term now has a contribution in the 5304 that cannot be balanced in the Bianchi identity, and so we conclude that the 27000 is not allowed.

Next, we consider the four-forms. They obey Bianchi identities of the form

$$dF^{\mathcal{X}}_4 = F^U_3 F^R_2 t_{RU}{}^{\mathcal{X}} \ , \tag{5.6.9}$$

where $t_{RU}{}^{\mathcal{X}}$ is an invariant tensor in the indicated representations. The possible representations are therefore contained in the tensor product of 248 and 3875 which is 779247+147250+30380+3875+248. In order for the Bianchi itself to be consistent we must have

$$b_{(RS}{}^U t_{T)U}{}^{\mathcal{X}} = 0 \tag{5.6.10}$$

This will be true if the symmetrised triple product of 248 does not contain the representation $\mathcal{X}$. Of the possible representations, only the 3875 and the 147250 have this property and so we can discard the others. However this argument does not rule out the 248 since a four form transforming under this representation can have the following Bianchi identity

$$dF^R_4 = F_3 F^R_2 + F^U_3 F^R_2 e_{RU}{}^R \ , \tag{5.6.11}$$

where $e_{RU}{}^R$ projects onto the 248 of $E8$. Using similar group-theoretical methods as above it is possible to show that also this Bianchi identity is consistent. There is also a possible singlet four-form F_4 but it has to be gauge-trivial, i.e. $dF_4 = 0$. However, this is trivial in the sense that one can write $F_4 = dG_3$, where the only non-zero component of G is $G_{abc} \propto \varepsilon_{abc}$.

Finally, we comment on the allowed five-forms that arise in the theory. The corresponding potentials for these do not have purely even components, but there are three-form gauge parameters that can have non-zero $(3,0)$-components. Indeed, as has been pointed out [78], these can play a rôle in the gauged theory. The five-form Bianchi identities have the schematic form

$$dF_5^{\mathcal{X}} = (F_4F_2)^{\mathcal{X}} + (F_3F_3)^{\mathcal{X}} \ . \tag{5.6.12}$$

The possible representations must therefore be contained in the following products

$$\begin{aligned}
248 \times 248 &= 1 + 3875 + 27000 + 30380 + 248 \\
3875 \times 248 &= 779247 + 147250 + 30380 + 3875 + 248 \\
147250 \times 248 &= 3875 + 30380 + 147250 + 779247 \\
&+ 2450240 + 6696000 + 26411008 \\
1 \times 3875 &= 3875 \\
(3875 \times 3875)_A &= 248 + 30380 + 779247 + 6696000 \ ,
\end{aligned} \tag{5.6.13}$$

where $_A$ stands for the antisymmetric product of the two 3875s. Of these only the five-forms transforming under the representations presented in table 2 have consitents Bianchi identities [126][3]. In table 2 we also summarise our results for the full set of allowed form fields and their multiplicity.

The only component of a five-form that can be non-zero in supergravity is $F_{3,2}$ and this must be proportional to $\varepsilon_{abc}\varepsilon_{\alpha\beta}$ multiplied by a function

[3] The representation content of this level has previously been determined in [40]

Form degree	Allowed forms	Form degree	Allowed Forms
2	**248**	4	**248 3875 147250**
3	**1 3875**	5	2 · **248** **3875** 2 · **30380** **147250 779247 6696000**

Table 5.1: The E8 representations and their multiplicity for forms of degree ≤ 5.

of the form $F_{ij}^{\mathcal{X}}$ that is antisymmetric on ij. It therefore follows that the only representations that can be non-zero must contain 120 in the branching down to $SO(16)$. This leaves only four possibilities: $248, 30380, 779247$ and 6696000. Given that their Bianchi identities are consistent we know from before that there will be no obstructions to solving them.

In summary, the dual two-forms are in the adjoint representation of E_8, the allowed three-forms transform under the singlet and 3875 representations and the allowed four-forms transform under the 248, 3875 and 147250 representations together with a trivial singlet. The Bianchi identities are

$$dF_2^R = 0 \tag{5.6.14}$$
$$dF_3 = F_2^S F_2^R a_{RS} \tag{5.6.15}$$
$$dF_3^U = F_2^R F_2^S b_{RS}{}^U \tag{5.6.16}$$
$$dF_4 = 0 \tag{5.6.17}$$
$$dF_4^R = F_3 F_2^R + F_3^U F_2^R e_{RU}{}^{\mathcal{X}} \tag{5.6.18}$$
$$dF_4^U = F_3^V F_2^R c_{RV}{}^U \tag{5.6.19}$$
$$dF_4^X = F_3^V F_2^R d_{RV}{}^X \ , \tag{5.6.20}$$

where a, b, c, d, e are E_8-invariant tensors, and where $X, Y, Z = 1, \ldots 147250$. The components of F_2^R are

$$\begin{aligned} F^R_{\alpha i \beta j} &= -2i\varepsilon_{\alpha\beta} V_{ij}{}^R \\ F^R_{a\beta j} &= -i(\gamma_a \Sigma_j \Lambda)_\beta{}^I V_I{}^R \\ F^R_{ab} &= \varepsilon_{ab}{}^c (P_c{}^I V_I{}^R - 2iA_c{}^{ij} V_{ij}{}^R) \end{aligned} \tag{5.6.21}$$

The components of the singlet F_3 are

$$\begin{aligned} F_{a\beta j\gamma k} &= -i\delta_{jk}(\gamma_a)_{\beta\gamma} \\ F_{ab\gamma k} &= 0 \\ F_{abc} &= 4i\varepsilon_{abc} B \ . \end{aligned} \tag{5.6.22}$$

The components of F_3^U are

$$\begin{aligned} F^U_{a\beta j\gamma k} &= -i(\gamma_a)_{\beta\gamma} V_{j,k}{}^U \\ F^U_{ab\gamma k} &= 2iV_{kI'}{}^U (\gamma_{ab}\Lambda)_{\gamma I'} \\ F^U_{abc} &= 2i\varepsilon_{abc} B^{ijkl} V_{ijkl}{}^U \ . \end{aligned} \tag{5.6.23}$$

The components of F_4^R are

$$\begin{aligned} F^R_{ab\gamma k\delta l} &= -i(\gamma_{ab})_{\gamma\delta} V_{k,l}{}^R \\ F^R_{abc\delta l} &= a\varepsilon_{abc}(\Sigma_l \Lambda)_\delta{}^I V_I{}^R \ , \end{aligned} \tag{5.6.24}$$

The components of F_4^U are

$$\begin{aligned} F^U_{ab\gamma k\delta l} &= -i(\gamma_{ab})_{\gamma\delta} V_{k,l}{}^U \\ F^U_{abc\delta l} &= a\varepsilon_{abc} V_{lI'}{}^U \Lambda_{\gamma I'} \ , \end{aligned} \tag{5.6.25}$$

while the components of F_4^X are

$$\begin{aligned} F^X_{ab\gamma k\delta l} &= -i(\gamma_{ab})_{\gamma\delta} V_{k,l}{}^X \\ F^X_{abc\delta l} &= b\varepsilon_{abc} V_{lI'}{}^X \Lambda_{\gamma I'} \ , \end{aligned} \tag{5.6.26}$$

where a, b are real, calculable constants. It is easy to see that the singlet four-form F_4 is exact as the only non-zero component is

$$F_{ab\gamma k\delta l} = -i\delta_{kl}(\gamma_{ab})_{\gamma\delta} \ . \tag{5.6.27}$$

Clearly $F_4 = dG_3$, where the only non-vanishing component of G_3 is $G_{abc} = \varepsilon_{abc}$.

In addition there can be non-zero five-forms in the representations 248,30380, 779247 and 6696000, obeying Bianchi identities of the form (5.6.12). The five-forms can only be non-vanishing at dimension zero where they have expressions of the form

$$F^{\mathcal{X}}_{abc\alpha i\beta j} = ic\varepsilon_{abc}\varepsilon_{\alpha\beta} V_{ij}{}^{\mathcal{X}} \ , \tag{5.6.28}$$

where $\mathcal{X}$ can be one of the above representations, ij denotes the 120 of $SO(16)$ and c is some real constant.

The forms can equally well be discussed in an $SO(16)$ basis. We shall distinguish this basis by barring quantities or indices. The Bianchi identities can be written

$$D\bar{F}_n = -\bar{F}_n \wedge P + \bar{F}_{n-1} \wedge \bar{F}_2 \tag{5.6.29}$$

where $\bar{F} = F\mathcal{V}$, P is considered as being Lie-algebra-valued in the appropriate representation (with barred indices) and where the last term is understood as involving the appropriate invariant tensor. For each F this equation can be split into various representations of $SO(16)$ according to the branching rules. The components of the Fs in this basis can be read off from those of the E_8 basis straightforwardly. A key point is that they do not contain any explicit scalars; in particular, the dimension-zero components are just given by $SO(16)$-invariant tensors.

5.7 N=8

In the half-maximal theory we give the Bianchi identities for all two, three, four and for a few of the five forms in the appendix B. The allowed form fields of degree ≥ 4 and their degeneracy are derivable from group theory alone, however we have verified this explicitly for all four forms and for five of the five forms. The possible form fields of degree ≤ 4 where first presented in [125] were the authors used a Kac-Moody approach while the five forms have not been determined previously. We present our results in table 3.

Below we give some examples of the components of the form fields of degree larger than space-time.

The F_4 transforming under the adjoint representation have two non-zero components

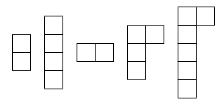
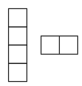
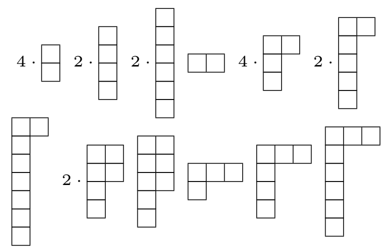
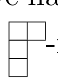
Form degree	Allowed forms	Form degree	Allowed Forms
2		4	
3	1 	5	

Table 5.2: The SO(8,n) representations and their multiplicity for forms of degree $\leq$ 5.

$$F_{4abc\alpha I}{}^{MN} \;=\; a\varepsilon_{abc}(\Sigma^i\Lambda)_I V_i{}^{MN}, \qquad (5.7.1)$$

while the only non-zero component of F_5 in the adjoint representation is

$$F_{5abc\delta I\epsilon J}{}^{MN} \;=\; -i\varepsilon_{abc}\varepsilon_{\delta\epsilon}\Sigma^{ij}{}_{IJ}V_{ij}{}^{MN}. \qquad (5.7.2)$$

As an explicit example of how we have determined the five forms we consider the one transforming under the -representation of $SO(8,n)$

$$
\begin{aligned}
dF_5{}^{[MNO],P} &= m(\bar{F}_4{}^{[MN}F_2{}^{O]P} + \bar{F}_4{}^{P[M}F_2{}^{NO]} + \frac{4}{6+n}\eta^{P[M}\bar{F}_4{}^{N}{}_{Q}F_2{}^{O]Q}) \\
&+ p(F_4{}^{P[M}F_2{}^{NO]} - \frac{2}{6+n}\eta^{P[M}F_4{}^{N}{}_{Q}F_2{}^{O]Q}) \\
&+ q(F_4{}^{MNO}{}_{Q}F_2{}^{PQ} + F_4{}^{P[MN}{}_{Q}F_2{}^{O]Q} \\
&- \frac{4}{6+n}\eta^{P[M}F_4{}^{NO]}{}_{QR}F_2{}^{QR}) \\
&+ r(F_4{}^{MNO,}{}_{Q}F_2{}^{PQ} + F_4{}^{P[MN,}{}_{Q}F_2{}^{O]Q} \\
&- \frac{4}{6+n}\eta^{P[M}F_4{}^{NO]}{}_{Q,R}F_2{}^{QR}) \\
&+ s(F_4{}^{[MN}{}_{Q,}{}^{O]}F_2{}^{PQ} + F_4{}^{P[M}{}_{Q,}{}^{N}F_2{}^{O]Q} \\
&+ \frac{1}{6+n}\eta^{P[M}(3F_4{}^{N}{}_{QR,}{}^{O]} + F_4{}^{NO]}{}_{Q,R})F_2{}^{QR}) \\
&+ t(F_4{}^{MNO}{}_{QR,}{}^{P}F_2{}^{QR} + F_4{}^{P[MN}{}_{QR,}{}^{O]}F_2{}^{QR}) \\
&+ u(F_3{}^{[MN}{}_{QR}F_3{}^{O]PQR} + \frac{2}{6+n}\eta^{P[M}F_3{}^{N}{}_{QRS}F_3{}^{O]QRS}) \\
&+ v(F_3{}^{MNO}{}_{V}F_3{}^{PV} - F_3{}^{P[MN}{}_{V}F_3{}^{O]V}) \qquad (5.7.3)
\end{aligned}
$$

where M, N, O, P are $SO(8,n)$ vector indices and $m, p, .., v$ are real constants. If the Bianchi identity is consistent the constants can be chosen such that $ddF_5{}^{[MNO],P} = 0$. If this can be done in n-ways the form is n-times degenerate. If the coefficients can not be chosen such that $ddF_5{}^{[MNO],P} = 0$ the Bianchi identity is inconsistent and the form field transforming under this representation will not be a part of the form field spectrum.

Taking the exterior derivative of (5.7.3) give rise to terms of the form $F_3{}^X \wedge F_2 \wedge F_2$ where X can be in one of the representations that the three forms transform under (these are listed in appendix B). The terms that are non-zero are those for which $\yng(2,1)$ is contained in the direct product $X \otimes \yng(1,1) \otimes \yng(1,1)$. The consistency of the Bianchi identity in hand give 5 equations involving the constants $m, p..., v$, they are soluble using the following constraints

$$
\begin{aligned}
r &= q - \frac{2}{3}u + \frac{6(2+n)}{5(4+n)}t \\
s &= -2q + 2u - \frac{6n}{5(4+n)}t \\
27m &= \frac{1}{3}(4+n)u + \frac{3}{5}(4+n)t + 2v \\
3p &= 4v - 4r + s.
\end{aligned}
\tag{5.7.4}
$$

Using the cohomological argument given previously, we know that there exists a solution to this Bianchi identity since it is consistent. There are 8 unknown constants in the Bianchi identity and 4 constraints. We can therefore conclude that a five form transforming under the $\ydiagram{2,1}$-representation of $SO(8,n)$ is allowed by supersymmetry and that it is 4 times degenerate.

Chapter 6

Gauged supergravity

The gauging of maximal $D = 3$ supergravity has been discussed in [76, 77], and the differential forms were subsequently discussed in [78]. The half-maximal theories in thee dimensions were first discussed in [79], [80] (see [119] for a discussion in various dimensions). The key tool in constructing gauged supergravity theories is the embedding tensor, $\mathcal{E}_R{}^S$.[1] The embedding tensor allows one to present the results in a way which looks covariant with respect to the duality group G but which is actually only covariant with respect to the local R-symmetry group and the gauge group $G_0 \subset G$ that we shall not need to specify explicitly (see [77] for a list of the possible gauge groups).

Gauging is very natural in superspace and the constraints on the embedding tensor can be seen as a direct consequence of the gauged Maurer-Cartan equation. Also, we note that in three dimensions gauging is special [76, 77, 78] because there are no independent vector degrees of freedom. We will construct the supergeometry of gauged supergravity by modifying the constraints at dimension one. For maximal supergravity in three dimensions

[1] The embedding tensor is usually called Θ but we have chosen a different notation to avoid confusion with the superspace coordinates.

we deduce all modifications of the supergeometry while we simply state how the set of constraints gets modified for $N = 8$ gauged supergravity. At the end of the chapter we will discuss the deformations of the forms fields in the maximal theory. This has previously been discussed in detail in [78]. Our approach is to deform the higher-rank fields in a covariant fashion, i.e. by deforming the Bianchi identities. Discussions of the E_{11} approach to gauged supergravity in $D = 3$ can be found in [71],[81].

6.1 Geometry of gauged supergravity

The embedding tensor is essentially given by a sum of projectors onto the irreducible subspaces of $\mathfrak{e}_8$ corresponding to the simple factors of $\mathfrak{g}_0$ [77]. It can be taken to be symmetric, $\mathcal{E}_{RS} := \mathcal{E}_R{}^T a_{TS} = \mathcal{E}_{SR}$ and there is also a quadratic constraint on $\mathcal{E}$ that follows from demanding that it be invariant under gauge transformations. It is

$$\mathcal{E}_R{}^P \mathcal{E}_{(M}{}^Q f_{N)PQ} = 0 \ , \qquad (6.1.1)$$

where f_{PQR} denotes the $\mathfrak{e}_8$ structure constants. The discussion is best approached via the gauged Maurer-Cartan form [87] (see [88] for the superspace version) which can be written

$$\Phi = \mathcal{D}\mathcal{V}\mathcal{V}^{-1} = P + Q \ , \qquad (6.1.2)$$

where $\mathcal{D}$ is a gauge-covariant derivative that acts on the E_8 index carried by $\mathcal{V}_{\bar{R}}{}^R$, i.e. the superscript. The gauged Maurer-Cartan equation, which follows directly from (6.1.2), is

$$R + DP + P^2 = g\mathcal{F} := g\mathcal{V}\mathcal{F}\mathcal{V}^{-1} \ . \qquad (6.1.3)$$

In $D = 3$ in the maximal theory, with E_8 duality group, the embedding tensor is a projector in the adjoint representation [76, 77], which in this case coincides with the fundamental. In the half-maximal case we can use a similar approach, that is, we can take the embedding tensor $\mathcal{E}_X{}^Y$ to be a projector in the adjoint representation, $X = [RS]$. This matrix, when the second index is lowered, is symmetric and projects onto the Lie algebra of the gauge group $\mathfrak{g}_0$.

g is a constant with dimensions of mass which characterises the deformation and D is covariant with respect to both the local R-symmetry group $SO(N)$ (in our cases $N = 16$ or 8) and G_0. The theory has both of these groups as local symmetries, but the duality group is broken. The technique we shall use in the following analysis is to work with $SO(N)$ indices, so that the gauge group is hidden from view.

The original geometrical constraint in superspace (3.1.1), i.e. taking the dimension-zero torsion to be the same as in flat space, together with the allowed conventional constraints, leads to the dimension-one torsion and curvatures given in equations (3.1.6) and (3.1.7). Since the deformation parameter g has dimension one it follows that we can expect changes to the tensors K_{ij}, L_{aij} and M_{ijkl}. These can only be proportional to g multiplied by functions of the scalars and so L_{aij} must be unchanged.

6.2 N=16

As mentioned in the previous section we expect changes to K and M. Functions occurring at dimension one in these superfields fall into the 1+135+1820 representations of $SO(16)$. Anticipating a little we can see that these can be combined into the E_8 representations $1+3875$ if we also have the $1920'$. This representation can be found in the scalar part of $D_{\alpha i}\Lambda_{\beta I'}$, which vanishes in the non-gauged case but which will be modified when the gauging is turned on as one can see from (6.1.3).

To implement the gauging explicitly we first need to solve for the two-form field strength. This should be projected along $\mathfrak{g}_0$ which leads us to propose that it should have the form

$$\mathcal{F}^R = F^S \mathcal{E}_S{}^R \ . \tag{6.2.1}$$

It is easy to see, using the fact that $\mathcal{D}\mathcal{E}_R{}^S = 0$, that the Bianchi identity for $\mathcal{F}^R$ will be solved if we take the components of F^R to have the same form as in the ungauged case. In fact, the only g-dependence could be at dimension one, but since this component of F is a spacetime two-form this cannot occur. Lowering the index on $\mathcal{F}$ and converting to an $SO(16)$ basis we find, at dimension zero,

$$\mathcal{F}_{\alpha i\beta j,kl} = -2i\varepsilon_{\alpha\beta} V_{ij}{}^R V^S_{kl}\, \mathcal{E}_{RS} \tag{6.2.2}$$

$$\mathcal{F}_{\alpha i\beta j,I} = -2i\varepsilon_{\alpha\beta} V_I{}^R V^S_{kl}\, \mathcal{E}_{RS} \ , \tag{6.2.3}$$

and at dimension one-half,

$$\mathcal{F}_{a\beta j,I} = -i(\gamma_a \Sigma_j \Lambda)_\beta{}^J V_J{}^R V^S_I\, \mathcal{E}_{RS} \ . \tag{6.2.4}$$

Since $\mathcal{E}_{RS}$ is symmetric it can contain the $1 + 3875 + 27000$ representations of E_8, but we can see directly from the 120 component of (6.1.3) that the 27000 must be absent due to the fact that it cannot be accommodated in the dimension-one curvature. This is the basic constraint on $\mathcal{E}$ derived in [76, 77]. It then follows that the only representation in (6.2.3) will be the 1920'. The functions appearing in the dimension-zero and one-half $\mathcal{F}_2$'s can therefore be written

$$
\begin{aligned}
V_{ij}{}^R V_{kl}^S \,\mathcal{E}_{RS} &= f_{ij,kl} := \delta_{i[k}\delta_{l]j} f_0 + \delta_{[i[k} f_{j],l]} + f_{ijkl} \\
V_I{}^R V_{kl}^S \mathcal{E}_{RS} &= (\Sigma_{[k})_{IJ'} f_{l]J'} \\
V_I{}^R V_J^S \,\mathcal{E}_{RS} &= \frac{1}{4!}(\Sigma_{ijkl})_{IJ} f^{ijkl} - 2\delta_{IJ} f_0 \;, \qquad (6.2.5)
\end{aligned}
$$

where the functions $f_0, f_{i,j}, f_{ijkl}$ and f_i exhibit the $1+135+1820+1920'$ split explicitly. The deformation feeds into the geometrical tensors at dimension one via the gauged Maurer-Cartan equation from which we find

$$
\begin{aligned}
K_{ij} &= -\frac{i}{2}\delta_{ij} B + 2g(f_{i,j} + \delta_{ij} f_0) \\
M_{ijkl} &= -iB_{ijkl} + 8g f_{ijkl} \\
D_{\alpha i}\Lambda_{\beta I'} &= (\gamma^a)_{\alpha\beta}(\Sigma_i P_a)_{I'} + g\varepsilon_{\alpha\beta} f_{iI'} \;. \qquad (6.2.6)
\end{aligned}
$$

It is easy to check that the geometrical Bianchi identities at dimension three-halves are satisfied. To do this one needs the following easily derivable identities

$$
\begin{aligned}
D_{\alpha i} f_0 &= 0 \\
D_{\alpha i} f_{j,k} &= 2i\Lambda_\alpha \Sigma_i \Sigma_{(j} f_{k)} \\
D_{\alpha i} f_{jklm} &= -i\Lambda_\alpha \Sigma_i \Sigma_{[jkl} f_{m]} \\
D_{\alpha i} f_j &= i\Lambda_\alpha \Sigma_i \Sigma^k f_{j,k} + \frac{i}{48}\Lambda_\alpha \Sigma_i (\Sigma_j{}^{klmn} f_{klmn} + 12\Sigma^{klm} f_{jklm}) \qquad (6.2.7)
\end{aligned}
$$

There is a modification to the gravitino field strength given by

$$
\Psi_{ai}(g) = 4g f_i \gamma_a \Lambda \;, \qquad (6.2.8)
$$

as well as a g-dependent term in the fermion equation of motion,

$$\gamma^a D_a \Lambda(g) = 4g(f_0 \Lambda - \frac{1}{48} f_{ijkl} \Sigma^{ijkl} \Lambda) \ . \tag{6.2.9}$$

Finally, at dimension two, there are changes to the curvature scalar and the scalar equation of motion given by

$$\begin{aligned}
R(g) &= -\frac{2ig}{3}(48 f_0 B - f_{ijkl} B^{ijkl}) \\
&\quad -3g^2(f_i f^i - 2 f_{i,j} f^{i,j} - 32 f_0 f_0) \\
D^a P_a(g) &= \frac{5ig}{4.4!} \Sigma_{ijk} f_l B^{ijkl} \\
&\quad -\frac{3g^2}{4}(\Sigma_i f_j f^{i,j} - \frac{1}{9} \Sigma_{jkl} f_i f^{ijkl}) \, .
\end{aligned} \tag{6.2.10}$$

6.3 N=8

The half-maximal theories have a local gauge group G_0, embedded in $G = SO(8,n)$. As in the maximal case we expect changes to K and M, in order to find these we first need to solve for the two-form field strength. This should be projected along $\mathfrak{g}_0$ which leads us to propose that it should have the form

$$\mathcal{F}^X = F^Y \mathcal{E}_Y{}^X \ . \tag{6.3.1}$$

It is easy to see, using the fact that $\mathcal{D}\mathcal{E}_X{}^Y = 0$, that the Bianchi identity for $\mathcal{F}^X$ will be solved if we take the components of F^X to have the same form as in the ungauged case. In fact, the only g-dependence could be at dimension one, but since this component of F is a spacetime two-form this cannot arise. At dimension one we therefore find

$$\mathcal{F}^{RS}_{\alpha I \beta J} = i\varepsilon_{\alpha\beta}(\Sigma^{ij})_{IJ} V_i^{\ T} V_j^{\ U} \mathcal{E}_{RS,TU} \ , \tag{6.3.2}$$

where we have replaced the adjoint indices on $\mathcal{E}$ by pairs of antisymmetrised vector indices. Using this and (6.1.3) we find that the deformations of the dimension-one geometrical tensors due to gauging have the form

$$\begin{aligned} R_{\alpha I \beta J,kl}(g) &= ig\varepsilon_{\alpha\beta}(\Sigma^{ij})_{IJ} f_{ij,kl} \\ R_{\alpha I \beta J,rs}(g) &= ig\varepsilon_{\alpha\beta}(\Sigma^{ij})_{IJ} f_{ij,rs} \\ D_{\alpha I}\Lambda_{\beta Jkr}(g) &= ig\varepsilon_{\alpha\beta}(\Sigma^{ij})_{IJ} f_{ij,kr} \ , \end{aligned} \tag{6.3.3}$$

where the functions f are defined by

$$f_{\bar{R}\bar{S},\bar{T}\bar{U}} := V_{\bar{R}}^{\ R} V_{\bar{S}}^{\ S} V_{\bar{T}}^{\ T} V_{\bar{U}}^{\ U} \mathcal{E}_{RS,TU} \ . \tag{6.3.4}$$

Since $\mathcal{E}_{XY}$ is symmetric the representations that it contains are four-index antisymmetric, two-index symmetric traceless, a singlet and a tensor with the symmetries of the Weyl tensor. In Young tableaux,

$$\left(\ydiagram{1,1} \otimes \ydiagram{1,1}\right)_{\text{sym}} = \ydiagram{1,1,1,1} \oplus \ydiagram{2} \oplus 1 \oplus \ydiagram{2,2} \tag{6.3.5}$$

If the Weyl tensor representation were non-zero, then there would be a contribution of the same type to $f_{ij,kl}$ which cannot be accommodated in M or K. So this representation must be absent, and there is therefore an extra constraint on $\mathcal{E}$. There are no problems with any of the other representations but there is an interesting point concerning the 35-dimensional representations that appear in $f_{ij,kl}$. In fact, all three can occur: the anti-self-dual four-form will deform M_{IJKL} while the self-dual four-form will deform the traceless part of K_{IJ}. The symmetric traceless 35 in $f_{ij,kl}$, then modifies the self-dual part of M_{IJKL}. Thus, in the generic gauged theory, it is not possible to impose the duality constraint on the superconformal multiplet.

Explicitly, we find that the deformations of the dimension-one scalar functions are given by

$$\begin{aligned} K_{IJ} &= ig(\delta_{IJ}f_0 + (\Sigma^{ijkl})_{IJ}f^{(+)}_{ijkl}) \\ M_{IJKL} &= ig(\Sigma^{ij})_{IJ}(\Sigma^{kl})_{KL}(f^{(-)}_{ijkl} + \delta_{ik}f_{jl}) \\ D_{\alpha I}\Lambda^{\alpha}_{J'r} &= ig((\Sigma^{ijk})_{IJ'}f_{ijkr} + (\Sigma^{i})_{IJ'}f_{ir})\ , \end{aligned} \tag{6.3.6}$$

where the functions on the right are in the irreducible representations indicated, with the plus and minus signs standing for self-dual and anti-self-dual respectively.

6.4 Hierarchy of forms

We now consider the hierarchy of forms. We will do this for the maximal case and we will see how the method of gauging a supergravity theory necessitates a tower of forms that in fact demands the existence of form fields of degree 5. In the geometrical discussion above we have only used the true non-abelian gauge fields, but in order to accommodate all of the forms it will be necessary to include the other two-form gauge fields which we could think of as being abelian, although they do transform under the gauge group. In other words we have a set of 248 gauge fields F^R, where $\mathcal{F}^R = F^S\mathcal{E}_S{}^R$. The Bianchis for the forms can then be written, in the E_8 basis,

$$\mathcal{D}F_n = (FF)_{n+1} + gF_{n+1}Y_{n+1,n} \ , \tag{6.4.1}$$

where (FF) denotes the bilinear term of the same form as in the ungauged case and $Y_{n+1,n}$ denotes a mapping from the representation space $\mathcal{R}_{n+1}$ of the $(n+1)$-forms to that of the n-forms, $\mathcal{R}_n$. In order to determine the Y-matrices one must compute the effect of applying $\mathcal{D}$ to (6.4.1); clearly one will require, in agreement with the general discussion in [73], that

$$Y_{n+1,n}\ Y_{n,n-1} = 0 \tag{6.4.2}$$

in order for the g^2 terms to cancel. The presence of the g-dependent term on the right-hand side of a deformed Bianchi identity implies that the $(n-1)$-form potential will transform under the $(n-1)$-form gauge transformation of the n-form potential, which is the way the hierarchy has been derived previously [78]. Before discussing this system in more detail we note that the Fs themselves are hardly changed from the abelian case. Since g has dimension one, it is only the purely even components of the Fs that can get deformed and these only by the f-functions of the previous section. So only the $(3,0)$ components of the three-forms can receive corrections, which for the 3875 take the form

$$F^U_{abc} = \varepsilon_{abc}(a' f_0 V_0{}^U + b' f^{i,j} V_{i,j}{}^U + c' f^{ijkl} V_{ijkl}{}^U) \ , \tag{6.4.3}$$

where $a', b'.c'$ are constants.

We now give an example of the hierarchy computation in this covariant language. The two-form Bianchi identity is $\mathcal{D}F_2 = gF_3Y_{3,2}$. Applying a second $\mathcal{D}$ to this we get

$$gF_2^T F_2^S X_{ST}{}^R = gF_2^S F_2^T b_{ST}{}^U Y_U{}^R + g^2 F_4^{\mathcal{X}} Y_{\mathcal{X}}{}^U Y_U{}^R \ , \tag{6.4.4}$$

where ${X_{ST}}^R = {\mathcal{E}_S}^P {f_{PT}}^R$ is the generator of the gauge group within the 248 representation [73].

The second term on the right must vanish in order to satisfy (6.4.2), and we can easily satisfy the part of the equation linear in g by taking

$$ {Y_{ST}}^R = {\mathcal{E}_{(S}}^P {f_{T)P}}^R \tag{6.4.5} $$

in agreement with [78]. Here, we have replaced U by a symmetrised pair of 248 indices on Y, and (6.4.5) is correct as it stands because the singlet and the 27000 vanish on the right. One can continue in this way for the higher forms for which we find $Y_{4,3}$s that agree with those of [78], although we have not checked beyond this. The complete hierarchy for the gauged forms in supergravity requires the five-forms and their Bianchi identities, and the Bianchi identities for the six-forms, even though the latter vanish in supergravity. This is because the seven-form right-hand side of the six-form Bianchi identities involve terms of the form (F_5, F_2) and (F_4, F_3) and these expressions can in principle be non-zero at dimension zero. However, provided that the identities are themselves consistent, these equations will automatically be satisfied for cohomological reasons.

The fact that the Bianchi identities are consistent suggest that the field strengths should be expressible in terms of potentials, and this is indeed the case. For the two- and three-forms we find

$$ F_2^R = dA_1^R + \frac{g}{2} A_1^T A_1^S {X_{ST}}^R + A_2^U {Y_U}^R \tag{6.4.6} $$

$$ F_3^{RS} = \mathcal{D}A_2^{RS} + g A_1^R (dA_1^S + \frac{1}{3} A_I^Q A_1^P {X_{PQ}}^S) + g A_4^{\mathcal{X}} {Y_{\mathcal{X}}}^{RS} \ , \tag{6.4.7} $$

where, in the second equation, the indices RS are symmetrised and projected onto the 3875, while $\mathcal{X}$ denotes the 3875 and the 147250 representations.

In the above we have ignored the singlet three-form, but its Bianchi identity can also be deformed:

$$dF_3 = F_2^R F_2^S a_{RS} + g F_4^U Y_U \ , \tag{6.4.8}$$

where Y_U is proportional to the 3875 component of the embedding matrix. It is not difficult to verify the consistency of this Bianchi identity.

Chapter 7

Introduction to Borcherds algebras

The form fields were first given an algebraic interpretation in [68] where a generator was associated to each potential such that the Maurer-Cartan equation for the sum of all field strengths generates the field equations. Two years later a correspondence between toroidal compactifications of M-theory and del Pezzo surfaces was found in [123]. Studying the cohomology of the del Pezzo surfaces the authors of [39] managed to extract the algebras found in [68]. These algebras are Borcherds algebras and a truncated set of their positive roots correspond to the generators of the potentials. The set of roots also contained information about the deformation and top form potentials.

There has also been much interest in decomposing over-extended algebras such as $E11$ to reach supergravity theories and it has even been suggested that the predictions these decompositions produce carry information beyond supergravity. Studying the form fields in a superspace setting leads in way to an opposite point of view. Instead of looking at the infinite Lie algebras as coming from some more fundamental theory, one can see that they are defined by the supergravity theories themselves. This is because, as we will see at the end of this chapter, the Bianchi identities define super Lie co-algebras. The algebras dual to these are Borcherds algebras of which we now give a brief introduction.

7.1 Borcherds algebras

The definition of a Borcherds (or generalised Kac-Moody) (super)-algebra starts with a generalised symmetric Cartan matrix, $(a_{ij}),\ i.j = 1 \dots N$, where some subset of the indices can be odd, which is non-degenerate and for which the following rules hold. The diagonal elements a_{ii} (no sum) can be positive, negative or zero, while the off-diagonal elements, $a_{ij},\ i \neq j$, are less or equal to zero. In the case that $a_{ii} > 0$, then $\frac{2a_{ij}}{a_{ii}} \in \mathbb{Z}, \forall j$, while if i is also odd $\frac{a_{ij}}{a_{ii}} \in \mathbb{Z}, \forall j$.

The Borcherds algebra $\mathcal{A}$ associated with (a_{ij}) is then determined by $3N$ generators $\{h_i, e_i, f_i\}$, $i = 1 \dots N$, satisfying the following conditions:

$$\begin{aligned} [h_i, h_j] &= 0 && (7.1.1) \\ [h_i, e_j] &= a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} e_j, \quad [e_i, f_j] = \delta_{ij} h_i && (7.1.2) \\ (\mathrm{ad}\, e_i)^{1-\frac{2a_{ij}}{a_{ii}}} e_j &= 0, \qquad \text{for } \ a_{ii} > 0 \text{ and } i \neq j && (7.1.3) \\ [e_i, e_j] &= 0 \qquad \text{when } \ a_{ij} = 0\ , && (7.1.4) \end{aligned}$$

with the last two conditions remaining valid if e_i, e_j are replaced by f_i, f_j. The generators h_i are even, and the generator f_i is even or odd if e_i is. If $a_{ii} > 0$ the integer $\frac{2a_{ij}}{a_{ii}}$ is negative, and if i is odd, it is also even.

In a Borcherds algebra there is still a triangular decomposition of the form $\mathcal{A} = \mathcal{N}^- \oplus \mathcal{H} \oplus \mathcal{N}^+$, and it is still possible to define roots as in the Kac-Moody case. Furthermore, if $a_{ii} > 0$, the algebra generated by $\{f_i, h_i, e_i\}$ for i even, or by these together with $[f_i, f_i]$ and $[e_i, e_i]$ when i is odd, are isomorphic to $\mathfrak{sl}(2)$ or $\mathfrak{osp}(1|2)$, respectively, and the algebra can be decomposed into finite dimensional representations of these (super)algebras. When $a_{ii} < 0$, one has the same algebras but the Borcherds algebra contains infinite-dimensional representations of them. In the case that $a_{ii} = 0$, the sub-algebra generated by $\{f_i, h_i, e_i\}$ is isomorphic to the Heisenberg (super)algebra.

The multiplicities of the roots for Borcherds algebras may be computed using the Peterson formula [121]. We found the discussion in [122] to be useful, especially for the super case. Let β be an element of the positive root lattice Q^+, i.e. a linear combination of the positive simple roots (in both of the cases we discuss in the text, α_0 and α_1,) with non-negative integral coefficients. And let $\mathfrak{g}_\beta$ be the subalgebra of the Borcherds algebra corresponding to a root β. The super-dimension of such a subalgebra is defined by $\mathrm{sdim}\, \mathfrak{g}_\beta = (-1)^{\deg \beta} \mathrm{dim}\, \mathfrak{g}_\beta$, where the degree is zero or one according to whether β is even or odd. The ordinary dimension is the multiplicity. The Peterson formula is

$$(\beta|\beta - 2\rho)c(\beta) = \sum (\beta'|\beta'')c(\beta')c(\beta'') \ , \tag{7.1.5}$$

where the sum is over all elements such that $\beta = \beta' + \beta''$, ρ is a special combination of the simple roots and the quantities $c(\beta)$ are determined by the formula

$$c(\beta) = \sum_n \frac{1}{n} \mathrm{sdim}(\frac{\beta}{n}) \ , \tag{7.1.6}$$

n being a positive integer. The quantity ρ is determined by requiring that the left-hand-side of (7.1.5) should be zero for the positive simple roots; for IIB, $\rho = -\alpha_0$ while for IIA, $\rho = 0$. The round brackets denote the scalar product determined by the Cartan matrix, with $(\alpha_i|\alpha_j) = a_{ij}$ for the positive simple roots. In the sum in (7.1.6) the dimension of an element β of Q^+ that is not a root is zero, although this does not mean that the corresponding

$c(\beta)$ vanishes because β may be a multiple of a root. Note that (7.1.6) only has more than one term if β is an integral multiple of a root. The $c(\beta)$s, and hence the multiplicities, can be computed from these two formulae in an iterative fashion.

7.2 Bianchi identities and Lie super co-algebras

In superspace the Bianchi identities define a Lie super co-algebra, to see this consider the $n+1$ form Bianchi identity

$$I_{\ell+2} = dF_{\ell+1} - \sum_{m+n=\ell} F_{m+1} \wedge F_{n+1}, \tag{7.2.1}$$

where we have assumed that the form fields carry a global symmetry index hidden from view. A Bianchi identity is consistent when I is closed. If a Bianchi identity is soluble then I is also exact. The consistent and soluble Bianchi identities can be written as

$$dF_{\ell+1} = \sum_{m+n=\ell} F_{m+1} \wedge F_{n+1}, \tag{7.2.2}$$

and they satisfy

$$d^2 F_{\ell+1} = \sum_{m+n=\ell} \big(F_{m+1} \wedge dF_{n+1} + (-1)^{n+1}(dF_{m+1}) \wedge F_{n+1}\big) = 0. \tag{7.2.3}$$

Equations (7.2.2) and (7.2.3) define a Lie super co-algebra where (7.2.3) is the equivalent of the super Jacobi identity. The set of all forms together with their consistent Bianchi identities constitutes a Lie super co-algebra. We recall that this is a $\mathbb{Z}_2$-graded vector space $\mathcal{A}$ together with a linear map $d : \mathcal{A} \to \wedge^2\mathcal{A}$ that squares to zero, where $\wedge$ denotes the graded antisymmetric tensor product. In our case there is also a $\mathbb{Z}$-grading,

$$\mathcal{A} = \oplus_{\ell \in \mathbb{Z},\, \ell \geq 1} \mathcal{A}_\ell = \mathcal{A}^+ \oplus \mathcal{A}^- \tag{7.2.4}$$

where $\mathcal{A}_\ell$ is the space of $(\ell + 1)$ field-strength forms, and where $\mathcal{A}^+$ and $\mathcal{A}^-$ denote the even and odd parts corresponding to ℓ even and ℓ odd respectively.

The Lie super algebra dual to this co-algebra is determined by the Bianchi identities which are not guaranteed to soluble by cohomology. The form fields of larger degree are determined by simply demanding their Bianchi identity to be closed. This however is equivalent of a super Jacobi identity and hence the form spectrum can be encoded by a Cartan matrix for some infinite algebra. The Cartan matrix is determined by solving the consistent and soluble Bianchi identities of a degree when the corresponding cohomomolgy group is not empty.

Chapter 8

Borcherds algebras

In the following chapter we show that the Borcherds algebras encoding the form fields of type IIA and IIB supergravity also predicts the form fields appearing in superspace beyond the space-time limit. This allows for a more natural interpretation of Borcherds algebras which are infinite while there are only finitely many form fields in the space-time approach. The latter will always lead to a truncated picture where most of the positive roots in the Borcherds algebra do not have an interpretation and have to be discarded.

We will also derive the Borcherds algebras of type IIA and IIB supergravity by analysing the Bianchi identities. This should be possible by the discussion in the previous chapter where we showed that the form fields automatically determine a Lie-(super)algebraic structure. This was seen to be so because they satisfy Bianchi identities of the form $dF = F^2$, which determines a coalgebra, while the consistency conditions for the Bianchi identities are equivalent to the Jacobi identity. This means that the consistent Bianchi identities, which are not guaranteed to be satisfied, determine a Cartan matrix of a Borcherds algebra which is the dual of the coalgebra. In a sense this is quite satisfying since it demystifies the appearance of infinite Lie algebras in supergravity. Adding to this the result of [44] one can explain why infinite algebras such as $E10$ [1] and $E11$ predicts the form field spectrum of maximal supergravity theories as will be explained at the end of this chapter.

We also identify the Borcherds algebras encoding the form-fields in half-maximal supergravity. This will allow us to use computer-based methods to identify all possible six-forms. In the final chapter we will provide some evidence that some of these are non-zero once string corrections are considered.

8.1 IIB

We begin to show that the Borcherds algebra for IIB also encodes the form fields beyond the spacetime limit. The Borcherds algebra for IIB is purely even since all of the field-strengths have odd degree. The Cartan matrix is

$$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} , \tag{8.1.1}$$

so that the fundamental commutation relations between the generators are

$$\begin{aligned} [h_0, e_0] &= 0 \qquad & [h_1, e_0] = -e_0 \\ [h_0, e_1] &= -e_1 \qquad & [h_1, e_1] = 2e_1 \\ [e_0, f_0] &= h_0 \qquad & [e_1, f_1] = h_1 \ , \end{aligned} \tag{8.1.2}$$

and we also have

$$(\mathrm{ad}\, e_1)^2 e_0 = 0 \ , \tag{8.1.3}$$

[1] $E10$ does not predict the top-form potentials which can also be understood by the discussion in the end of the chapter

while $\{f_1, h_1, e_1\}$ forms a basis for $\mathfrak{sl}(2)$. The vectors e_0, e_1 are eigenvectors associated with the positive simple roots, α_0, α_1, respectively.

It is clear from the discussion of the previous section that the algebra of forms is generated from the three-form field strengths. We shall associate a generator with each potential form, so the three-form generators will be denoted e^2_R, the five-forms by e^4 and so on. In IIB these are all even generators. We write the sum of all the field-strengths as $\mathbb{F} = \sum(F^X_n e^{n-1}_X)$ where X denotes the appropriate representation of $\mathfrak{sl}(2)$, so that all of the Bianchi identities can be combined into

$$d\mathbb{F} = \frac{1}{2}[\mathbb{F}, \mathbb{F}] \ , \tag{8.1.4}$$

where the commutator denotes the commutator of the basis elements, and where due care has to be taken with signs. The generators e^2_R form an $\mathfrak{sl}(2)$ doublet, so if we identify the lowest weight e^2_1 with e_0, the second one can be obtained from it by the raising operator e_1, so $e^2_2 = [e_0, e_1] := e_{01}$. This cannot be raised any further so that we have the relation (8.1.3), $(\mathrm{ad}\, e_1)^2 e_0 = 0$. To make further progress we investigate some of the states that are generated. For F_5 and F^R_7 we have

$$\begin{aligned} e^4 &= [e_0, e_{01}] \\ e^6_1 &= [e_0, [e_0, e_{01}]] \\ e^6_2 &= [e_{01}, [e_0, e_{01}]] \ . \end{aligned} \tag{8.1.5}$$

Continuing in this way we find, for form degree $2n+1$, a state of the form $(\mathrm{ad}\, e_0)^{n-1} e_{01}$, and this series can increase without limit. Moreover, it is clear that each state will be characterised by a corresponding root, although one should be aware that these can occur with multiplicities. Each $(2n+1)$-form is associated with the roots $\alpha = n\alpha_0 + m\alpha_1$ where $m = 1, \dots (n-1)$, although there can be multiplicities starting from $n = 5$. It is easy to see that one recovers the previously known results up to level 5, i.e. the forms of degree eleven that saturate the spacetime limit (ten-form potentials). However, the positive roots have been tabulated beyond this level [109], so that we can easily compare our results from section three to the Borcherds prediction up to level 7, i.e. fifteen-forms. The table is as follows:

Level	Form degree	$\mathfrak{sl}(2)$ representation(s)
1	3	2
2	5	1
3	7	2
4	9	3
5	11	4+2
6	13	5+3+1
7	15	6+4(2)+2(2)

The form degrees here are those of the field strengths while the figures in brackets in the last entry in the third column indicate that these representations appear with multiplicity two. Comparing with the Bianchi identities in section 5.4 we find exact agreement including the correct multiplicities for the fifteen-forms. It is also easy to see that the roots are correctly given. For example, the vector $(\mathrm{ad}\, e_0)^n e_1$ is the lowest weight state of the largest representation at level n corresponding to the root vector $n\alpha_0 + \alpha_1$. Indeed, this result is not surprising because it is clear that the algebra of forms must be isomorphic to the positive root algebra $\mathcal{N}^+$ modulo the one-dimensional space generated by e_1.

It is also clear that the Borcherds algebra determined by (8.1.2) is the smallest Borcherds algebra that can accommodate the IIB form algebra. The existence of an $sl(2)$ subalgebra implies that $a_{11} = 2$ while (8.1.3) tells us that $a_{01} = -1$ (and hence, by symmetry, that $a_{10} = -1$). The fact that one can have arbitrary powers of $ad\, e_0$ means that a_{00} cannot be positive. If it was negative there would be a second $sl(2)$ subalgebra with infinite-dimensional representations within the Borcherds algebra, but this is not possible because there is only a finite number of forms of a given degree. So $a_{00} = 0$ and we are thus led to the Cartan matrix (8.1.2).

8.2 IIA

The Cartan matrix for IIA is given by

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} . \tag{8.2.1}$$

The super-algebra has generators $\{f_0, f_1, h_0, h_1, e_0, e_1\}$, where e_1, f_1 are odd, which obey the basic commutation relations

$$\begin{aligned} [h_0, e_0] &= 0 \\ [h_0, e_1] &= -1 \\ [h_1, e_0] &= -1 \\ [h_1, e_1] &= 0 . \end{aligned} \tag{8.2.2}$$

Since $a_{00} = a_{11} = 0$, the subalgebras associated with both sets of generators are of Heisenberg type.

The form algebra is generated from G_2 and H_3 so we shall associate elements of this algebra with them. For G_2 this is the odd element, e_1, while for H_3 it is the even element e_0. The first point to notice is that the Bianchi identity

$$dH_3 = G_2 G_2 \tag{8.2.3}$$

is consistent, since $dG_2 = 0$, and so, from the form point of view, we must have the relation $[e_1, e_1] = 0$. But this is also required from the general rules for a Borcherds superalgebra in the previous chapter.

For the RR forms the situation is very simple. G_4 is associated with $[e_0, e_1] := e_{01}$, G_6 with $[e_0, e_{01}] := e_{001}$ and so on. For G_{2n} the element of the algebra is $(\mathrm{ad}\, e_0)^{n-1} e_1$, and this series can increase without limit since $a_{00} = 0$.

For the NS forms the situation is slightly more complicated, but can be obtained directly from the Bianchi identities. For the seven-form one has only one possibility, namely $[e_{01}, e_{01}]$, while for the nine-form one has $[e_0, [e_{01}, e_{01}]]$. However, for the eleven-forms one finds two possibilities, $[e_0, [e_0, [e_{01}, e_{01}]]]$ and $[e_{001}, e_{001}]$. (Note that all of the vectors $e_{00..1}$ for any number of zeroes are odd.) For the thirteen-forms there are again two possibilities, $(\mathrm{ad}\, e_0)^3 [e_{01}, e_{01}]$ and $\mathrm{ad}\, e_0 [e_{001}, e_{001}]$. These results are in agreement with those of section three. This pattern continues to higher levels, so that there is a series of terms of this type obtained by acting with $\mathrm{ad}\, e_0$ on vectors of the form $[(\mathrm{ad}\, e_0)^k e_1, (\mathrm{ad}\, e_0)^k e_1]$. The situation can be summarised rather simply. For the RR forms, G_{2n}, $n \geq 1$, the roots are $(n-1)\alpha_0 + \alpha_1$, all with multiplicity one. For the NS forms, one has H_3 with root α_0 and two series (both with $n \geq 1$): the $(4n+3)$-forms, which correspond to the roots $2n\alpha_0 + 2\alpha_1$, and which have multiplicity n, and the $(4n+5)$-forms, which correspond to the roots $(2n+1)\alpha_0 + 2\alpha_1$ and which also have multiplicity n.

In the IIA case the algebra of forms is clearly isomorphic to the positive part, $\mathcal{N}^+$, of the Borcherds superalgebra. It is also clear that this Borcherds superalgebra is the smallest one that can accommodate the form algebra. Since $[e_1, e_1] = 0$ it follows that $a_{11} = 0$ while a_{00} must be zero for similar reasons to the IIB case. It is then possible to normalise the generators so that $a_{01} = a_{10} = -1$.

8.3 N=8

To find the Lie super algebra dual to the co-algebra of half-maximal supergravity in three dimensions we recall the techniques used in [42] to decompose $E10$ in a level-by-level expansion. The form fields in supergravity should be in one-to-one correspondence with the positive roots of an infinite Lie super algebra. We will make the assumption that a positive root of the Lie super algebra can be written as

$$\alpha = l\alpha_0 + \sum m^j \alpha_j. \tag{8.3.1}$$

Here l/ m^j are positive integers denoting the number of times the simple root α_0/α_j appears in α, we refer to the number l as the level of the root. We label the generators associated to the simple roots $\pm\alpha_0$ and $\pm\alpha_i$ by e_0/f_0 and e_i/f_i. We will take the generators e_i and f_i to be the generators of the duality group for reasons that will soon become clear. We also refer to the level of a generator according to the number of times e_0 appears in its commutator. The adjoint action of e_i on a generator does not alter its level so that all generators at a given level transform under a direct sum of representations the duality group.

The representations appearing at level $l+1$ are contained in the product $r_l \otimes r_1$, where r_l denotes the representations appearing at level l, however this will not be true for all representations. Of the representations that do appear there will be one generator associated to its highest weight. This is most easily seen if one considers the generator f^Λ corresponding to the negative root $-\alpha$. If $\mathrm{ad}e_i(f^\Lambda) = 0$, then f^Λ acts as a highest weight state for one of the representations appearing at level l. The weight of the state f^Λ is $h_i(f^\Lambda) = p_i$, where p_i is the Dynkin label for the representation. All generators that do not correspond to highest weight states are derivable from these by acting on f^Λ by $\mathrm{ad}f_i$.

Determining the Cartan matrix A for the Lie super algebra is rather trivial under the above assumption. A will be completely specified by analysing the Bianchi identities for the two and three form Bianchi identities. The first step to note is that all generators appearing at each level must, in order for the duality to work, transform under representations of $SO(8,n)$. If A has the form

$$A = \begin{pmatrix} A_{00} & A_{0i} \\ A_{i0} & A_{ij} \end{pmatrix},$$

as suggested by (8.3.1) then A_{ij} is the Cartan matrix for $SO(8,n)$. To determine A_{0i} we note that f_0 acts as a highest weight state for $SO(8,n)$ at level one. The weight of this state is $h_i(f_0) = -A_{i0} = p_i$. The two-form field strength is in the adjoint representation with Dynkin labels (010...0), hence we demand the generators at level one to transform under the same representation. We can therefore conclude that $A_{0i} = (0,-1,0,...,0)$ and without loss of generality we can take $A_{i0} = A_{0i}$.

To determine A_{00} we will match the representations at level two in the roots to those appearing in the three-forms. The generators at level two are formed by commuting the generators at level one, and the representations that can appear at level two are therefore contained in the symmetric product

$$\left(\ydiagram{1,1} \otimes \ydiagram{1,1}\right)_S = \ydiagram{2,2} + \ydiagram{1,1,1,1} + \ydiagram{2} + 1. \qquad (8.3.2)$$

A_{00} could take the values ≤ -1, 0 or 2; the roots corresponding to these values were given the following Dynkin diagrams in [39]

○	Bosonic real root of length 2
⊗	Bosonic imaginary root of length ≤ 0
●	Fermionic "imaginary" root of length 1, $A_{00} = 0$
◉	Fermionic imaginary root of length $\leq$ -1

We will discuss the different nodes in turn. If $A_{00} \leq -1$ then $[f_0, f_0]$ is a generator at level two. Moreover, it would be a highest weight state of $SO(8,n)$ since $\mathrm{ad}e_i([f_0, f_0]) = 0$. This generator would therefore give rise to the Weyl representation appearing at level two with weight $h_i([f_0, f_0]) = (020...0)$. Going back to the form fields we see that the Weyl representation is not allowed by supersymmetry so we cannot choose $A_{00} = -1$. If $A_{00} = 2$ only the adjoint representation appears at level 3. Hence we are left with $A_{00} = 0$. There are two type of nodes with length 0, bosonic or fermionic however e_0 need to be fermionic to reflect that the two forms commute, leaving us with the following Borcherds algebras.

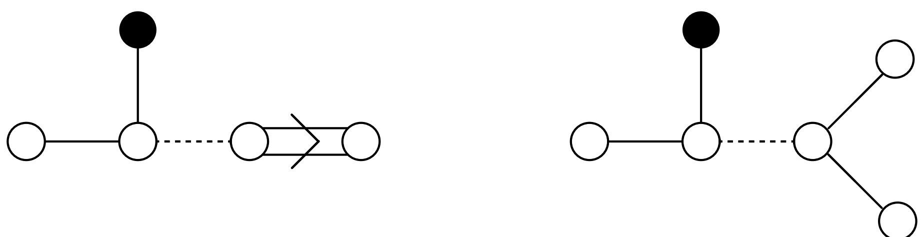


Figure 1. The Dynkin diagrams of the Borcherds algebras encoding the form field strengths

The Dynkin diagrams in figure 1 correspond to the Borcherds algebras that encode the form field spectrum. When the duality group is $SO(8, 2n-1)$ the diagram to the left is relevant, while if the duality group is $SO(8, 2n)$ the Dynkin diagram to the right should be used. We have verified that the above Borcherds algebras do indeed reproduce the representations in table 2. We have done this by using a generalisation of a result from [72] that the p-form spectrum of E^{+++} is a truncated Borcherds algebra. The generalization given in [44] states that the level decomposition with respect to a fermionic simple root of length zero in a Borcherds algebra can be obtained by replacing the corresponding black node with an infinite chain of white nodes, corresponding to bosonic simple roots of length 2. The upshot of this is that we can use a computer program [124] to calculate the representations up to any level by adding appropriately many white nodes to a B_n or D_n diagram to find the representation content at each level. The method of decoding the information of the Cartan matrices defined by the Dynkin diagrams in figure 1 is thus equivalent to the way the authors of [125] found the allowed form fields, the difference being that one adds more white nodes if one is interested in form fields of higher degree. The modules that do not go with totally antisymmetric tensors are not defined by the Bianchi identities. From this point of view they are objects appearing when one extracts the representation content at each level from the Borcherds algebra using a Kac-Moody algebra.

The above analysis is not limited to the supergravity discussed in this chapter nor to the duality groups we have encountered. It will be the case that Borcherds algebras are defined by the Bianchi identities in many other supergravity theories.

Chapter 9

Corrections at order α'

In the presence of corrections of order α' some higher-degree forms can in principle have non-zero components. Forms with bi-degrees $(p, 6)$ can have contributions of the form α' times scalars, while $(p, 5)$-forms can have contributions linear in Λ multiplied by α'. Here we shall focus on the latter as they are slightly easier to discuss. In principle this could be affected by neglecting the former, but for the non-trivial example to be discussed below it will turn out that there can be no such contribution. To simplify things we shall also consider only the case $n = 1$, i.e. the duality group is $SO(8, 1)$.

The Bianchi identities we need to consider have the form

$$dF_n = F_2 \wedge F_{n-1} + \dots \ , \tag{9.0.1}$$

where $n = p+5$ for $p = 0, 1, 2, 3$. In particular we shall focus on the case $p = 1$ and make the assumption that $F_{0,6} = 0$. The lowest non-trivial component of (9.0.1) that we are interested in has the form

$$t_0 \overset{(1)}{F}_{1,5} = \overset{(0)}{F}_{0,2} \overset{(1)}{F}_{0,5} \ , \tag{9.0.2}$$

since no other terms can contribute at order α'. Here, the superscripts indicate the order of α' in the given terms. As a first example, let us consider the case when the six-form is in the adjoint representation. The Bianchi identity is

$$dF_6^{RS} = F_2^{T[R} F_5^{S]}{}_T + \dots \ . \tag{9.0.3}$$

We can now use the scalar matrix to rewrite this equation in an $SO(8) \times SO(1)$ basis. For the term we are interested in this will be valid provided that there is no scalar contribution in $F_{0,6}$. We therefore find a term

$$t_0 F^i_{1,5} = F^{ij}_{0,2} F_{0,5\,j} \ , \tag{9.0.4}$$

where here, and below, we shall omit the order superscripts as it should be clear from the context which ones are meant. This is the only term that can appear on the right because $F_{0,5}$ has an odd number of unprimed $Spin(8)$ indices, so that we need an odd number of external vector indices in order to be able to find a linear Λ term. The F_2 term is

$$F^{ij}_{\alpha I \beta J} = i\varepsilon_{\alpha\beta}(\Sigma^{ij})_{IJ} \ . \tag{9.0.5}$$

The $F_{0,5}$ term must contain the spinor $\Lambda^{I'}_\alpha$, and since the five odd indices are totally symmetric, it follows that the $Spin(8)$ indices must be in the Young tableau arrangement $\ydiagram{3,2}$. This decomposes into the following representations

$$\ydiagram{3,2} = (0210) + (0030) + (0110) + (0010) \ . \tag{9.0.6}$$

We need to multiply these by the additional vector index, or (1000) and then look for possible (0001)s which could correspond to the spinor field Λ.

There is just one possibility and that comes from the (0010) representation in (9.0.6). Before we examine the right-hand side it is necessary to check whether this possibility is trivial in the sense that it can be removed by a field redefinition of the potential $A^i_{1,3}$. Consider the sequence

$$\Omega^i_{2,1} \xrightarrow{t_0} \Omega^i_{1,3} \xrightarrow{t_0} \Omega^i_{0,5} \ , \tag{9.0.7}$$

where $\Omega^i_{p,q}$ denotes the space of (p,q)-forms with an additional vector index i. If the element we are interested in is the image of t_0 acting on $\Omega^i_{1,3}$ then it can be removed by a field redefinition. Now there is just one possible Λ term in $\Omega^i_{2,1}$, namely

$$(\gamma_{ab}\Sigma^i\Lambda))_{\alpha I} \ ,$$

while there are two possible Λ terms in $\Omega^i_{1,3}$, as one can see by a little group theoretical analysis. (In this case there are two possible arrangements of the Lorentz spinor indices due to the additional Lorentz vector index, so the $Spin(8)$ indices can be in the tableaux $\ydiagram{2,1}$ or $\ydiagram{3}$.) One of these must therefore be t_0 exact, so the second one must map to $\Omega^i_{0,5}$. If this were not the case, this element would have to be in the cohomology group $H^{1,3}_t$, but this is zero. The conclusion of this analysis is that there are no non-trivial Λ terms in the six-forms in the adjoint representation.

It turns out that a similar situation obtains for the six-forms in the smallest representations of $SO(8,1)$, i.e. ((0000), (0100), (1000), (2000) and (0010), so that the first representation that can provide a non-trivial solution is in the four-form representation of $SO(8,1)$, i.e. (0002).

The Bianchi identity is

$$dF_6^{MNPQ} = F_2^{RS} F_5^{MNPQ,}{}_{RS} + \dots \ , \tag{9.0.8}$$

where F_5 on the right is in the $\yng(2,2,1,1)$ representation. Projecting onto $SO(8)$ indices, we find that there are two possible $SO(8)$ representations for F_5 on the right-hand side that can contain Λ given by the tableaux $\yng(2,2,1)$ and $\yng(2,1,1,1)$, or (0111) and either (1020) or (1002) in terms of $SO(8)$ Dynkin labels. It turns out that both the latter cannot contain any non-trivial Λ terms and so can be discarded. The relevant term in the Bianchi identity is therefore

$$t_0 F_{1,5}^{ijk} = F_{0,2}^{lm} F_{0,5}^{ijk,}{}_{lm} \tag{9.0.9}$$

The analysis goes in the same way as the previous example. The $Spin(8)$ indices on $F_{0,5}$ are again in the $\yng(3,2)$ tableau, while the additional $SO(8)$ indices are in the representation (0111). We find there are two possible Λ terms but that one of them is t_0 exact and so can be removed by a field redefinition. So the question is whether this term, when multiplied by $F_{0,2}$, becomes t_0 exact. To answer this consider the sequence

$$\Omega_{3,1}^{ijk} \xrightarrow{t_0} \Omega_{2,3}^{ijk} \xrightarrow{t_0} \Omega_{1,5}^{ijk} \xrightarrow{t_0} \Omega_{0,7}^{ijk} \; . \tag{9.0.10}$$

It is straightforward to find the number of possible Λ terms that can occur in each space. We find $1, 3, 4$ and 2 such terms in each space starting from the left. Since there is no t_0 cohomology except perhaps for $H_t^{0,7}$, we can immediately see that there can be two non-trivial Λ terms in $F_{1,5}^{ijk}$ and therefore both of the Λ terms in the $(0,7)$ form, $J_{0,7}^{ijk}$ say, are in fact in the image of t_0. In other words, $H_t^{0,7,ijk}$ restricted to the representation (0001) vanishes. As we have seen there are two possible Λ terms in $F_{0,5}^{ijk,}{}_{lm}$ and these give rise to the two Λ terms in $J_{0,7}^{ijk}$. In fact, the $Spin(8)$ spinor indices for $J_{0,7}^{ijk}(\Lambda)$ must be in the tableau $\yng(4,3)$ which can be rewritten as

$$\yng(4,3) = \yng(3,2) + (0310) + (0130) \; . \tag{9.0.11}$$

When one tensors this with the representation (0011) one finds that the last two representations cannot give rise to a Λ, whereas the first gives rise to two possibilities of this type. Clearly these correspond to the two we have identified earlier in $F_{0,5}^{ijk,}{}_{lm}$. So there is a single non-trivial solution to this Bianchi identity.

It is not difficult to see that this conclusion cannot be affected by a possible scalar term in $F_{0,6}$ on the left. This could give a term of the form $d_1 F_{0,6}^{MNPQ}$, projected onto an $SO(8)$ basis. In order to have a Lorentz scalar in $F_{0,6}$ the $Spin(8)$ indices would have to be in the $\ydiagram{3,3}$ tableau, but the four $SO(8,1)$ indices, which are totally antisymmetric, give rise to at least three antisymmetrised indices when broken down to $SO(8)$ representations and therefore a scalar term cannot be accommodated because they would need to be contracted with three antisymmetrised indices coming from the odd form indices.

Chapter 10

Conclusions

In the superspaces which extend $2 + 1$ dimensional space-time there is a constraint on the dimension-zero torsion tensor which leads to an off-shell superconformal geometry valid for any number of supersymmetry generators. As such it serves as a starting point for studying a wide range of three-dimensional supersymmetric theories.

We first used this formalism to study superconformal matter models and their coupling to conformal supergravity. We re-derived the well-known results in [33] and [34], [35], stating that the allowed gauge groups for $N = 6$ and $N = 8$ superconformal matter are $U(N) \times U(N)$ and $SU(2) \times SU(2)$ respectively. When coupling these matter multiplets to conformal supergravity, we found that $N = 8$ superconformal matter can also be charged under $SO(N)$. These models however are a bit unusual since they do not have a flat limit. They do admit AdS solutions although their interpretations in terms of branes is not clear.

Starting from the same off-shell superconformal geometry, we obtained half-maximal and maximal Poincaré supergravity by introducing further constraints. The curvature associated to the local $SO(N)$ symmetry is not only defined in terms of the $SO(N)$ connection, but also by the Maurer-Cartan equation in the associated non-linear sigma model. The sigma model fields therefore enter the geometry via the dimension one components of the torsion and curvature tensors. We solved for the geometry at dimension one in the half-maximal case with sigma models of the form $(SO(8)\times SO(n))\backslash SO(8,n)$, and for the complete geometry in the maximal theory, where the scalar fields live in the coset $SO(16)\backslash E8$. Using the Ricci identity, we also derived the equations of motion for the scalar and fermion fields in the latter theory.

The Poincaré supergravity theories admit a set of differential forms that transform in various representations of the duality group. We derived these sets in the above theories, as well as in type IIA and IIB supergravity, using only supersymmetry and duality. The analysis is made easier in superspace owing to the fact that one can study the problem covariantly. This is even possible for the D-form potentials, because a $D+1$-form makes sense in superspace since odd basis forms commute. In addition, explicitly checking that the Bianchi identities are satisfied is facilitated by the use of superspace cohomology; one only need to insure that a few components of the Bianchi identities for some of the lower degree-forms are satisfied. The use of superspace cohomology then insures that the Bianchi identities for all higher degree forms are satisfied, provided they are consistent.

We then went on to study gaugings of three-dimensional Poincaré supergravity theories. This can be done by introducing a non-abelian gauged subgroup of the duality group and making use of the gauged Maurer-Cartan form. There are terms in the gauged-deformed Maurer-Cartan equation involving the two-form gauge field strength that are proportional to the parameter g and that modify the dimension-one scalar functions in the theory. In the maximal case we computed the changes induced in all the components of the geometrical tensors and the modifications to the equations of motion. The dimension-one functions fall into the representations $1+135+1820+1920'$ of $SO(16)$, which are the representations that appear in the $1+3875$ representations of E_8. This gives a nice derivation of the fact that the embedding tensor, which is in the symmetric product of two adjoint representations of E_8, cannot contain the 27000 representation.

The differential forms can also be studied in the gauged theory by deforming the Bianchi identities. The field-strengths transform under the gauge group so that the exterior derivative must be made gauge-covariant. The Bianchi identity for F_n develops a term $gF_{n+1}Y_{n+1,n}$, where $Y_{n+1,n}$ maps the representation space of $(n+1)$-forms to that of n-forms and depends linearly on the embedding tensor. This means that all of the forms become related by a sequence of such maps that must be exact in order for the Bianchi identities to be consistent. The closure of the full system of forms requires the presence of $D+2$-form field strengths in the supergravity limit.

The form fields automatically determine a Lie-(super)algebraic structure because they satisfy Bianchi identities of the form $dF = F^2$, which determines a coalgebra, while the consistency conditions for the Bianchis are equivalent to the Jacobi identity. We showed that the consistent Bianchi identities, which are not guaranteed to be satisfied, determine a Cartan matrix of a Borcherds algebra which is the dual of the coalgebra. The Cartan matrix can then be used to generate the entire form field spectrum. This can be easily understood since the only restriction to the existence of the higher degree forms [1] are that they satisfy the equivalent of a Jacobi identity. We derived the previously known Borcherds algebras of IIA and IIB supergravity and also the Borcherds algebras encoding the form fields in the half-maximal supergravity theories in three dimensions. This analysis is not limited to the theories discussed here, but applies to all supergravity theories with sigma models of the form $H\backslash G$ where G is a semi-simple Lie Group and H its maximal compact sub-group. Adding to this the result of [44], which relates certain type Kac-Moody algebras to Borcherds algebras, this provides an explanation for why $E11$, $E10$ and the Kac-Moody approach to half-maximal supergravity

[1] By higher degree forms, we mean those which have Bianchi identities that are guaranteed to be satisfied by a cohomology argument.

theories predicts the p-form spectrum correctly.

Finally we discussed the meaning of Borcherds algebras when string corrections are taken into account. The Borcherds algebras not only encode the physical forms, their duals, and the form fields of the deformation and top-form potentials, but also an infinite number of higher degree forms. Even though all the form fields of degree larger than $D+2$ are identically zero in supergravity, there are is no need to truncate the algebra, and it is still possible to analyse which are the allowed representations using supersymmetry and duality. In type IIB supergravity we showed that the allowed representations for the thirteen- and fifteen-forms agree with those encoded by the associated Borcherds algebra, and similarly for type IIA supergravity. On dimensional grounds there is nothing to prevent these forms becoming non-zero in the presence of string corrections, but this is not in general an easy problem to investigate. In $D = 10$, for example, one could have α'^3 corrections in the $(0, 13)$ components of the thirteen-forms that would have to be linear in the dilatinos. The problem is that one is faced with representations of the spin group involving the tensor product of thirteen spinors. On the other hand, in half-maximal supergravity in three dimensions there can be non-zero six-forms whose lowest components would have to be α' multiplied by dimension one-half functions of the fields. We showed that the Bianchi identity for such a six-form is satisfied, but to show that it is non-zero in the presence of α'-corrections one should principle go back to the beginning and solve all of the Bianchi identities sequentially.

Chapter 11

Acknowledgments

First of all I would like to give my warmest thanks to my supervisor Paul Howe for being truly inspiring, endlessly helpful and knowledgeable and altogether the most kind hearted person.

I would like to thank Nordita for their hospitality. My final year was very much facilitated thanks to the people working there. I am grateful to Jakob Palmkvist and Université Libre de Bruxelles and also to Bengt Nilsson and Chalmers University of Technology for their invitations and hospitality. I also thank Jakob Palmkvist, Bengt Nilsson and Ulf Gran for stimulating collaborations.

For financial support I would like to thank Tekn. dr Marcus Wallenbergs Stiftelse and the STFC.

Appendix A

Lie algebra conventions

A.1 Conventions for $SO(8)$ and $SO(8,n)$

$SO(8)$ vector indices are $i, j, \ldots = 1 \ldots 8$, unprimed Weyl spinor indices are $I, J, \ldots = 1 \ldots 8$ and primed Weyl spinor indices are $I', J', \ldots = 1 \ldots 8$. These correspond to the representations (1000), (0010) and (0001), respectively. The metrics for each three spaces are flat euclidean, so it is not important to distinguish between upper and lower indices.

The basic sigma-matrices are $(\Sigma_i)_{IJ'}$ and $(\tilde{\Sigma}_i)_{J'I}$. We shall take $\tilde{\Sigma}_i = (\Sigma_i)^T$ and not bother to write out the tildes since it will be clear from the context which is meant. Sigma-matrices with two or more indices are antisymmetrised products of the basic ones as usual.

Sigma-matrices with an even number of vector indices are bi-spinors of a fixed chirality. Σ_2 give a basis of antisymmetric 8×8 matrices while $(1, \Sigma_4)$ give basis of symmetric matrices. We shall take $(\Sigma_{i_1 \ldots i_4})_{IJ}$ to be self-dual while Σ_8 with primed indices is anti-self-dual.

For an arbitrary matrix M_{IJ} we have

$$M_{IJ} = \frac{1}{8}\sum_{n=0}^{n=2}(\Sigma^{i_1\ldots i_{2n}})_{IJ}M_{i_1\ldots i_{2n}}\ , \tag{A.1.1}$$

where

$$M_{i_1\ldots i_{2n}} := \frac{1}{(2n)!}(\Sigma_{i_1\ldots i_{2n}})^{IJ}M_{IJ}\ , \tag{A.1.2}$$

except for $n = 4$ when there is an extra factor of $\frac{1}{2}$ on the right-hand side. The matrix Σ_0 is δ_{IJ}. The formula for primed indices is identical.

The bilinears that can be formed from the spinor field $\Lambda^r_{\alpha I'}$ in the text the Lorentz scalars

$$\begin{aligned} B &= \Lambda\Lambda := \Lambda^{\alpha I' r}\Lambda_{\alpha I' r} \\ B_{i_1\ldots i_4} &= \Lambda\Sigma_{i_1\ldots i_4}\Lambda := \Lambda^{\alpha I' r}(\Sigma_{i_1\ldots i_4})_{I'J'}\Lambda_{\alpha J' r}\ , \end{aligned} \tag{A.1.3}$$

and the spacetime vectors

$$A_{aij} = \Lambda\Sigma_{i_1 i_2}\gamma_a\Lambda := \Lambda^{\alpha I' r}(\Sigma_{ij})_{I'J'}(\gamma_a)_\alpha{}^\beta\Lambda_{\beta J' r} \tag{A.1.4}$$

Vector indices for $SO(8,n)$ are denoted by R, S, etc, while those for $SO(n)$ are r, s, etc. Indices for the adjoint representation are denoted by X, Y, etc, so that $X = [RS]$. The metric is $\eta_{RS} = (\delta_{ij}, -\delta_{rs})$, and indices are raised and lowered using this metric, including η_{rs} for $SO(n)$ indices.

A.2 Conventions for $SO(16)$ and E_8

Vector indices are $i, j, \ldots = 1 \ldots 16$, unprimed Weyl spinor indices are $I, J, \ldots = 1 \ldots 128$ and primed Weyl spinor indices are $I', J', \ldots = 1 \ldots 128$. The metrics for each three spaces are flat euclidean, so it is not important to distinguish between upper and lower indices.

The basic sigma-matrices are $(\Sigma_i)_{IJ'}$ and $(\tilde{\Sigma}_i)_{J'I}$. We shall take $\tilde{\Sigma}_i = (\Sigma_i)^T$ and not bother to write out the tildes since it will be clear from the context which is meant. Sigma-matrices with two or more indices are antisymmetrised products of the basic ones as usual.

Sigma-matrices with an even number of vector indices are bi-spinors of a fixed chirality. (Σ_2, Σ_6) give a basis of antisymmetric 128×128 matrices while $(1, \Sigma_4, \Sigma_8)$ give a basis of symmetric matrices. We shall take $(\Sigma_{i_1 \ldots i_8})_{IJ}$ to be self-dual while Σ_8 with primed indices is anti-self-dual.

For an arbitrary matrix M_{IJ} we have

$$M_{IJ} = \frac{1}{128} \sum_{n=0}^{n=4} (\Sigma^{i_1 \ldots i_{2n}})_{IJ} M_{i_1 \ldots i_{2n}} \ , \tag{A.2.1}$$

where

$$M_{i_1 \ldots i_{2n}} := \frac{1}{(2n)!} (\Sigma_{i_1 \ldots i_{2n}})^{IJ} M_{IJ} \ , \tag{A.2.2}$$

except for $n = 4$ when there is an extra factor of $\frac{1}{2}$ on the right-hand side. The matrix Σ_0 is δ_{IJ}. The formula for primed indices is identical.

The bilinears that can be formed from the spinor field $\Lambda_{\alpha I'}$ are the Lorentz scalars

$$\begin{aligned} B &= \Lambda\Lambda := \Lambda^{\alpha I'}\Lambda_{\alpha I'} \\ B_{i_1\dots i_4} &= \Lambda\Sigma_{i_1\dots i_4}\Lambda := \Lambda^{\alpha I'}(\Sigma_{i_1\dots i_4})_{I'J'}\Lambda_{\alpha J'} \\ B_{i_1\dots i_8} &= \Lambda\Sigma_{i_1\dots i_8}\Lambda := \Lambda^{\alpha I'}(\Sigma_{i_1\dots i_8})_{I'J'}\Lambda_{\alpha J'} \ , \end{aligned} \tag{A.2.3}$$

and the spacetime vectors

$$\begin{aligned} A_{ai_1i_2} &= \Lambda\Sigma_{i_1i_2}\gamma_a\Lambda := \Lambda^{\alpha I'}(\Sigma_{i_1i_2})_{I'J'}(\gamma_a)_\alpha{}^\beta\Lambda_{\beta J'} \\ A_{ai_1\dots i_6} &= \Lambda\Sigma_{i_1\dots i_6}\gamma_a\Lambda := \Lambda^{\alpha I'}(\Sigma_{i_1\dots i_6})_{I'J'}(\gamma_a)_\alpha{}^\beta\Lambda_{\beta J'} \ . \end{aligned} \tag{A.2.4}$$

The adjoint representation of E_8 is the same as the defining representation and has dimension 248. It splits into 120+128 in $SO(16)$. The summation convention is $A^R B_R = A^I B_I + \frac{1}{2}A^{ij}B_{ij}$. The metric a_{RS} has components

$$\begin{aligned} a_{IJ} &= \delta_{IJ} \\ a_{ij,kl} &= -\frac{1}{2}\delta_{k[i}\delta_{j]l} \ . \end{aligned} \tag{A.2.5}$$

For the inverse, the summation convention implies that $a^{IJ} = \delta^{IJ}$ while

$$a^{ij,kl} = -8\delta^{k[i}\delta^{j]l} \ . \tag{A.2.6}$$

Appendix B

Bianchi identities

B.1 Two Forms

$$dF_2{}^{[MN]} = 0 \tag{B.1.1}$$

B.2 Three Forms

$$\begin{aligned}
dF_3 &= F_{2OP}F_2{}^{OP} \\
dF_3{}^{(MN)} &= F_2{}^M{}_Q F_2{}^{NQ} - \frac{1}{8+n}F_{2OP}F_2{}^{OP} \\
dF_3{}^{[MNOP]} &= F_2{}^{[MN}F_2{}^{OP]}
\end{aligned} \tag{B.2.1}$$

B.3 Four Forms

$$
\begin{aligned}
d\bar{F_4}^{[MN]} &= F_3F_2{}^{MN} + \frac{3}{7}(6F_3{}^{[M}{}_Q F_2{}^{N]Q} \\
&- 9BF_3{}^{MN}{}_{OP}F_2{}^{OP}) \\
dF_4{}^{(MN)} &= F_3{}^{(M}{}_Q F_2{}^{N)Q} \\
dF_4{}^{[MNOP]} &= F_3{}^{[MNO}{}_Q F_2{}^{P]Q} \\
dF_4{}^{[MNO],P} &= (F_3{}^{P[M}F_2{}^{NO]} - \frac{2}{6+n}\eta^{P[M}F_3{}^N{}_Q F_2{}^{O]Q}) \\
&+ \frac{3}{4}(F_3{}^{MNO}{}_Q F_2{}^{PQ} + F_3{}^{P[MN}{}_Q F_2{}^{O]Q} - \frac{4}{6+n}\eta^{P[M}F_3{}^{NO]}{}_{QR}F_2{}^{QR}) \\
dF_4{}^{[MNOPQ],R} &= F_3{}^{[MNOP}F_2{}^{Q]R} + F_3{}^{R[MNO}F_2{}^{PQ]} + \frac{6}{4+n}\eta^{R[M}F_3{}^{NOP}{}_V F_2{}^{Q]V}
\end{aligned}
\tag{B.3.1}
$$

B.4 Five Forms

$$
\begin{aligned}
d\bar{F_5}^{[MN]} &= a\bar{F_4}^{[M}{}_Q F_2{}^{N]Q} + bF_4{}^{[M}{}_Q F_2{}^{N]Q} \\
&+ cF_4{}^{MN}{}_{PQ}F_2{}^{PQ} + hF_4{}^{MN}{}_{P,Q}F_2{}^{PQ} \\
&+ fF_3{}^M{}_Q F_3{}^{NQ} + gF_3{}^M{}_{PQR}F_3{}^{NPQR}
\end{aligned}
\tag{B.4.1}
$$

Constraints

$$
\begin{aligned}
-\frac{54}{6+n}a - 3b - \frac{4(5+n)}{6+n}h + 6f &= 0 \\
-\frac{162}{6+n}a + 3c + \frac{3(10+n)}{6+n)}h + 6g &= 0
\end{aligned}
\tag{B.4.2}
$$

$$
\begin{aligned}
dF_5{}^{[PQRSTU]} &= hF_4{}^{[PQRS}F_2{}^{TU]} + iF_3{}^{[PQR}{}_V F_3{}^{STU]V} \\
&+ j(F_4{}^{[PQRST,}{}_V F_2{}^{UV]} + F_{4V}{}^{[PQRS,T}F_2{}^{U]V})
\end{aligned} \quad \text{(B.4.3)}
$$

Constraints

$$
h + 2i - \frac{10+n}{5(5+n)}j = 0 \quad \text{(B.4.4)}
$$

$$
\begin{aligned}
dF_5{}^{[MNO],P} &= o(\bar{F}_4{}^{[MN}F_2{}^{O]P} + \bar{F}_4{}^{P[M}F_2{}^{NO]} \\
&+ \frac{4}{6+n}\eta^{P[M}\bar{F}_4{}^{N}{}_Q F_2{}^{O]Q}) \\
&+ p(F_4{}^{P[M}F_2{}^{NO]} - \frac{2}{6+n}\eta^{P[M}F_4{}^{N}{}_Q F_2{}^{O]Q}) \\
&+ q(F_4{}^{MNO}{}_Q F_2{}^{PQ} + F_4{}^{P[MN}{}_Q F_2{}^{O]Q} \\
&- \frac{4}{6+n}\eta^{P[M}F_4{}^{NO]}{}_{QR}F_2{}^{QR}) \\
&+ r(F_4{}^{MNO,}{}_Q F_2{}^{PQ} + F_4{}^{P[MN,}{}_Q F_2{}^{O]Q} \\
&- \frac{4}{6+n}\eta^{P[M}F_4{}^{NO]}{}_{Q,R}F_2{}^{QR}) \\
&+ s(F_4{}^{[MN}{}_{Q,}{}^{O]}F_2{}^{PQ} + F_4{}^{P[M}{}_{Q,}{}^{N}F_2{}^{O]Q} \\
&+ \frac{1}{6+n}\eta^{P[M}(3F_4{}^{N}{}_{QR,}{}^{O]} + F_4{}^{NO]}{}_{Q,R})F_2{}^{QR}) \\
&+ t(F_4{}^{MNO}{}_{QR,}{}^{P}F_2{}^{QR} + F_4{}^{P[MN}{}_{QR,}{}^{O]}F_2{}^{QR}) \\
&+ u(F_3{}^{[MN}{}_{QR}F_3{}^{O]PQR} + \frac{2}{6+n}\eta^{P[M}F_3{}^{N}{}_{QRS}F_3{}^{O]QRS}) \\
&+ v(F_3{}^{MNO}{}_V F_3{}^{PV} - F_3{}^{P[MN}{}_V F_3{}^{O]V})
\end{aligned} \quad \text{(B.4.5)}
$$

Constraints

$$\begin{aligned}
r &= q - \frac{2}{3}u + \frac{6(2+n)}{5(4+n)}t \\
s &= -2q + 2u - \frac{6n}{5(4+n)}t \\
27o &= \frac{1}{3}(4+n)u + \frac{3}{5}(4+n)t + 2v \\
3k &= 4v - 4r + s.
\end{aligned} \tag{B.4.6}$$

$$\begin{aligned}
dF_5{}^{(MNO),P} &= p(F_4{}^{(MN}F_2{}^{O)P} \\
&- \frac{1}{8+n}(2\eta^{P(M}F_4{}^{N}{}_{Q}F_2{}^{O)Q} \\
&- \eta^{(MN}(F_4{}^{O)}{}_{Q}F_2{}^{PQ} + F_4{}^{|P|}{}_{Q}F_2{}^{O)Q}))) \\
&+ q(F_4{}^{P}{}_{Q}{}^{(M,N}{}_{Q}F_2{}^{O)Q} \\
&- \frac{1}{2(8+n)}(2\eta^{P(M}F_{4QR}{}^{N,O)}F_2{}^{QR} \\
&- \eta^{(MN}(F_{4QR}{}^{O),P}F_2{}^{QR} + F_{4QR}{}^{|P|,O)}F_2{}^{QR}))) \\
&+ r(F_3{}^{(MN}F_3{}^{O)P} - \frac{1}{2(6+n)}F_3{}^{P[MN}{}_{V}F_3{}^{O]V})
\end{aligned} \tag{B.4.7}$$

Constraints

$$p = r = -\frac{q}{3} \tag{B.4.8}$$

$$
\begin{aligned}
dF_5{}^{[MNOP],}{}_{[QR]} &= \Delta(F_4^{MNOP}F_{2QR} + F_{4QR}{}^{[MN}F_2{}^{OP]} + 2F_4{}^{[MNO}{}_{[Q}F_2{}^{P]}{}_{R]} \\
&- \frac{10}{4+n}\eta^{[M}_{[Q}(F_4{}^{NOP]}{}_TF_{2R]}{}^T + F_{4R]}{}^{NO}{}_TF_2{}^{P]T}) \\
&+ \frac{20}{(4+n)(5+n)}\eta^{[M}_Q\eta^N_R F_4{}^{OP]}{}_{TS}F_2{}^{TS}) \\
&+ \Gamma((F_4{}^{[MNO,}{}_{[Q} + F_{4[Q}{}^{MN,O})F_2{}^{P]}{}_{R]} \\
&+ (F_4{}^{[MN}{}_{[Q,R]} - F_{4QR}{}^{[M,N})F_2{}^{OP]} \\
&- \frac{1}{(4+n)}\eta^{[M}_{[Q}((F_4{}^{NOP]}{}_T + F_4{}^{[NO}{}_{T,}{}^{P]})F_{2R]}{}^T \\
&- (5F_4{}^{NO}{}_{|T|,R]} + 6F_4{}^N{}_{R]T,}{}^O - F_4{}^{NO}{}_{R],T}))F_2{}^{P]T} \\
&+ \frac{2}{3(4+n)}\eta^{[M}_Q\eta^N_R(F_4{}^{OP]}{}_{U,V} - F_{4UV}{}^{O,P]})F_2{}^{QR}) \\
&+ \Lambda((F_4{}^{MNOP}{}_{V,[Q} + F_4{}^{[MNO}{}_{[Q|V|,}{}^{P]})F_{2R]}{}^V \\
&+ (F_4{}^{[MN}{}_{QRV,}{}^O - F_4{}^{[MNO}{}_{[Q|V|,R]})F_2{}^{P]V} \\
&- \eta^{[M}_{[Q}(F_4{}^{NO}{}_{R]UV,}{}^{P]} + F_4{}^{NOP]}{}_{UV,R)}F_2{}^{UV}) \\
&+ \Sigma(F_3{}^{[MNO}{}_TF_3{}^{P]}{}_{QR}{}^T - F_3{}^{[MN}{}_{TQ}F_3{}^{OP]T}{}_R \\
&+ \frac{10}{(4+n)}\eta^{[M}_{[Q}F_3{}^{NO}{}_{TS}F_3{}^{P]}{}_{R]}{}^{TS} \\
&+ \frac{10}{(4+n)(5+n)}\eta^{[M}_Q\eta^N_R F_3{}^O{}_{TSU}F_3{}^{P]TSU})
\end{aligned}
\tag{B.4.9}
$$

Constraints

$$
\begin{aligned}
-\Gamma - \frac{4}{3}\Sigma - \frac{3}{2}\Delta + \frac{3}{10}\frac{(10+n)}{4+n}\Lambda &= 0 \\
\Gamma + \Sigma + \frac{1}{2}\Delta - \frac{3}{10}\frac{(10+2n)}{4+n}\Lambda &= 0
\end{aligned}
\tag{B.4.10}
$$

Appendix C

N=16 D=1 supergravity

In this appendix we give the superspace geometry of $D = 1$ $N = 16$ Poincaré supergravity with sigma model of the form $SO(16)\backslash E8$ [126]. The structure and conventions are the same as for the on-shell maximal Poincaré supergravity in three dimensions discussed in the main text (except that the Lorentz group is trivial). We stress that this is an off-shell formulation.

The physical scalars and fermions are contained in the one-form field $P_A = (P_{i,I}, P_{0,I})$ where

$$\begin{aligned} P_{i,I} &= i(\Sigma_i)_{IJ'}\Lambda^{J'} \\ P_{0,I} &= \frac{1}{16}(\Sigma^i)_{IJ'}D_i\Lambda^{J'} \ . \end{aligned} \tag{C.0.1}$$

The supersymmetry variations of the superfield Λ are

$$D_i\Lambda^{I'} = (\Sigma_i)^{I'K}P_{0,K} \tag{C.0.2}$$

$$D_{[i}D_{j]}\Lambda^{J'} = (\Sigma_{[i}\Sigma^k\Lambda)^{J'}\Lambda\Sigma_{j]k}\Lambda - i(\Sigma_{ij}D_0\Lambda)^{J'} \ . \tag{C.0.3}$$

The non-vanishing components of the torsion are

$$T_{ij}{}^{0} = -2i\delta_{ij} \tag{C.0.4}$$

$$T_{0ij} = i\Lambda\Sigma_{ij}\Lambda\ . \tag{C.0.5}$$

The non-vanishing components of the curvature are

$$R_{ij,kl} = 2\Lambda(4\delta_{(i[k}\Sigma_{j)l]} - \delta_{ij}\Sigma_{kl}))\Lambda \tag{C.0.6}$$

$$R_{i0,kl} = -ia\Lambda(\Sigma_i)(\Sigma_{kl})(\Sigma_j)D^j\Lambda\ . \tag{C.0.7}$$

Bibliography

[1] S. R. Coleman and J. Mandula, “All Possible Symmetries Of The S Matrix,” Phys. Rev. **159** (1967) 1251.

[2] D. V. Volkov and V. P. Akulov, “Is the Neutrino a Goldstone Particle?,” Phys. Lett. B **46** (1973) 109.

[3] J. Wess and B. Zumino, “Supergauge Transformations in Four-Dimensions,” Nucl. Phys. B **70** (1974) 39.

[4] L. Brink, J. H. Schwarz and J. Scherk, “Supersymmetric Yang-Mills Theories,” Nucl. Phys. B **121** (1977) 77.

[5] S. Mandelstam, Nucl. Phys. B **213** (1983) 149.

[6] L. Brink, O. Lindgren and B. E. W. Nilsson, “The Ultraviolet Finiteness of the N=4 Yang-Mills Theory,” Phys. Lett. B **123** (1983) 323.

[7] P. S. Howe, K. S. Stelle and P. K. Townsend, Nucl. Phys. B **214** (1983) 519.

[8] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, “Progress Toward a Theory of Supergravity,” Phys. Rev. D **13** (1976) 3214.

[9] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, “Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes,” Phys. Rev. D **85** (2012) 105014 [arXiv:1201.5366 [hep-th]].

[10] R. E. Kallosh, Phys. Lett. B **99** (1981) 122.

[11] P. S. Howe and U. Lindstrom, Nucl. Phys. B **181** (1981) 487.

[12] H. Elvang and M. Kiermaier, JHEP **1010** (2010) 108 [arXiv:1007.4813 [hep-th]].

[13] Z. Bern, L. J. Dixon and R. Roiban, "Is N = 8 supergravity ultraviolet finite?," Phys. Lett. B **644** (2007) 265 [hep-th/0611086].

[14] R. Kallosh, "The Ultraviolet Finiteness of N=8 Supergravity," JHEP **1012** (2010) 009 [arXiv:1009.1135 [hep-th]].

[15] O. Klein, "Quantum Theory and Five-Dimensional Theory of Relativity. (In German and English)," Z. Phys. **37** (1926) 895 [Surveys High Energ. Phys. **5** (1986) 241].

[16] T. Kaluza, "On the Problem of Unity in Physics," Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921** (1921) 966.

[17] E. Cremmer, B. Julia and J. Scherk, "Supergravity Theory in Eleven-Dimensions," Phys. Lett. B **76** (1978) 409.

[18] E. Cremmer and B. Julia, "The SO(8) Supergravity," Nucl. Phys. B **159** (1979) 141.

[19] M. B. Green and J. H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory," Phys. Lett. B **149** (1984) 117.

[20] C. M. Hull and P. K. Townsend, "Unity of superstring dualities," Nucl. Phys. B **438** (1995) 109 [hep-th/9410167].

[21] H. Leutwyler, "A (2+1)-dimensional model for the quantum theory of gravity," Nuovo Cim. **42** (1966) 159.

[22] S. Deser, R. Jackiw and G. 't Hooft, Annals Phys. **152** (1984) 220.

[23] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69** (1992) 1849 [hep-th/9204099].

[24] A. Achucarro and P. K. Townsend, Phys. Lett. B **180** (1986) 89.

[25] E. Witten, Nucl. Phys. B **311** (1988) 46.

[26] S. Deser, R. Jackiw and S. Templeton, Annals Phys. **140** (1982) 372 [Erratum-ibid. **185** (1988) 406] [Annals Phys. **185** (1988) 406] [Annals Phys. **281** (2000) 409].

[27] W. Li, W. Song and A. Strominger, JHEP **0804** (2008) 082 [arXiv:0801.4566 [hep-th]].

[28] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. **102** (2009) 201301 [arXiv:0901.1766 [hep-th]].

[29] E. A. Bergshoeff, O. Hohm, J. Rosseel, E. Sezgin and P. K. Townsend, Class. Quant. Grav. **28** (2011) 015002 [arXiv:1005.3952 [hep-th]].

[30] P. S. Howe and R. W. Tucker, J. Math. Phys. **19** (1978) 869.

[31] P. S. Howe and R. W. Tucker, J. Phys. A A **10** (1977) L155.

[32] M. Brown and S. J. Gates, Jr., Annals Phys. **122** (1979) 443.

[31], [30], [32]

[33] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," JHEP **0810** (2008) 091 [arXiv:0806.1218 [hep-th]].

[34] J. Bagger and N. Lambert, "Modeling Multiple M2's," Phys. Rev. D **75** (2007) 045020 [hep-th/0611108].

[35] A. Gustavsson, "Algebraic structures on parallel M2-branes," Nucl. Phys. B **811** (2009) 66 [arXiv:0709.1260 [hep-th]].

[36] B. Julia, "Kac-moody Symmetry Of Gravitation And Supergravity Theories," LPTENS-82-22.

[37] S. 'y. Mizoguchi, Nucl. Phys. B **528** (1998) 238 [hep-th/9703160].

[38] P. C. West, Class. Quant. Grav. **18** (2001) 4443 [hep-th/0104081].

[39] P. Henry-Labordere, B. Julia and L. Paulot, "Borcherds symmetries in M-theory," JHEP **0204** (2002) 049 [arXiv:hep-th/0203070].

[40] D. S. Berman, M. Cederwall, A. Kleinschmidt and D. C. Thompson, arXiv:1208.5884 [hep-th].

[41] C. M. Hull, JHEP **0707** (2007) 079 [hep-th/0701203].

[42] T. Damour, M. Henneaux and H. Nicolai "E10 and a "small tension expansion" of M Theory" Phys. Rev. Lett. **89** (2002) 221601 [arXiv:hep-th/0207267v1].

[43] F. Riccioni, D. Steele and P. West, "The E(11) origin of all maximal supergravities - the hierarchy of field-strengths," JHEP **0909** (2009) 095 [arXiv:0906.1177 [hep-th]].

[44] J. Palmkvist, "Borcherds and Kac-Moody extensions of simple finite-dimensional Lie algebras," arXiv:1203.5107 [hep-th].

[45] A. Staruszkiewicz, "Gravitation theory in three-dimensional space," Acta Phys. Polon. **24** (1963) 735.

[46] B. Binegar, "Relativistic Field Theories In Three-Dimensions," J. Math. Phys. **23** (1982) 1511.

[47] L. Brink, A. M. Khan, P. Ramond and X. -z. Xiong, J. Math. Phys. **43** (2002) 6279 [hep-th/0205145].

[48] O. Hohm and J. Louis, Class. Quant. Grav. **21** (2004) 4607 [hep-th/0403128].

[49] B. de Wit, A. K. Tollsten and H. Nicolai, Nucl. Phys. B **392** (1993) 3 [hep-th/9208074].

[50] E. Cremmer and B. Julia, "The SO(8) Supergravity," Nucl. Phys. B **159** (1979) 141.

[51] P. van Nieuwenhuizen, Phys. Rev. D **32** (1985) 872.

[52] M. Rocek and P. van Nieuwenhuizen, Class. Quant. Grav. **3** (1986) 43.

[53] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, "Gauge Theory of the Conformal and Superconformal Group," Phys. Lett. B **69** (1977) 304.

[54] E. Bergshoeff, M. de Roo and B. de Wit, "Extended Conformal Supergravity," Nucl. Phys. B **182** (1981) 173.

[55] E. S. Fradkin and A. A. Tseytlin, "Conformal Anomaly in Weyl Theory and Anomaly Free Superconformal Theories," Phys. Lett. B **134** (1984) 187.

[56] E. S. Fradkin and A. A. Tseytlin, "Conformal Supergravity," Phys. Rept. **119** (1985) 233.

[57] P. S. Howe, "A Superspace Approach To Extended Conformal Supergravity," Phys. Lett. B **100** (1981) 389.

[58] J. Lott, "The Geometry of supergravity torsion constraints," math/0108125 [math-dg].

[59] S. J. Gates, Jr., K. S. Stelle and P. C. West, "Algebraic Origins Of Superspace Constraints In Supergravity," Nucl. Phys. B **169** (1980) 347.

[60] N. Dragon, "Torsion And Curvature In Extended Supergravity," Z. Phys. C **2** (1979) 29.

[61] P. S. Howe, "Weyl superspace," Phys. Lett. B **415** (1997) 149 [hep-th/9707184].

[62] D. S. Berman, L. C. Tadrowski and D. C. Thompson, Nucl. Phys. B **802** (2008) 106 [arXiv:0803.3611 [hep-th]].

[63] U. Gran, J. Greitz, P. Howe and B. E. W. Nilsson, arXiv:1204.2521 [hep-th].

[64] H. Nicolai and N. P. Warner, "The structure of N=16 supergravity in two dimensions," Commun. Math. Phys. **125** (1989) 369.

[65] A. Kleinschmidt and H. Nicolai, "E(10) and SO(9,9) invariant supergravity," JHEP **0407** (2004) 041 [arXiv:hep-th/0407101].

[66] N. Marcus and J. H. Schwarz, "Three-Dimensional Supergravity Theories," Nucl. Phys. B **228** (1983) 145.

[67] E. Cremmer, B. Julia, H. Lu and C. N. Pope, "Dualisation of dualities. I," Nucl. Phys. B **523** (1998) 73 [arXiv:hep-th/9710119].

[68] E. Cremmer, B. Julia, H. Lu and C. N. Pope, "Dualisation of dualities. II: Twisted self-duality of doubled fields and superdualities," Nucl. Phys. B **535** (1998) 242 [arXiv:hep-th/9806106].

[69] B. L. Julia, "Dualities in the classical supergravity limits: Dualisations, dualities and a detour via 4k+2 dimensions," arXiv:hep-th/9805083.

[70] F. Riccioni and P. C. West, "The E(11) origin of all maximal supergravities," JHEP **0707** (2007) 063 [arXiv:0705.0752 [hep-th]].

[71] E. A. Bergshoeff, I. De Baetselier and T. A. Nutma, "E(11) and the embedding tensor," JHEP **0709** (2007) 047 [arXiv:0705.1304 [hep-th]].

[72] M. Henneaux, B. L. Julia and J. Levie, "E_{11}, Borcherds algebras and maximal supergravity," arXiv:1007.5241 [hep-th].

[73] B. de Wit and H. Samtleben, "The end of the p-form hierarchy," JHEP **0808** (2008) 015 [arXiv:0805.4767 [hep-th]].

[74] E. Bergshoeff, P. S. Howe, S. Kerstan and L. Wulff, "Kappa-symmetric SL(2,R) covariant D-brane actions," JHEP **0710** (2007) 050 [arXiv:0708.2722 [hep-th]].

[75] E. A. Bergshoeff, J. Hartong, P. S. Howe, T. Ortin and F. Riccioni, "IIA/IIB Supergravity and Ten-forms," JHEP **1005** (2010) 061 [arXiv:1004.1348 [hep-th]].

[76] H. Nicolai and H. Samtleben, "Maximal gauged supergravity in three dimensions," Phys. Rev. Lett. **86** (2001) 1686 [arXiv:hep-th/0010076].

[77] H. Nicolai and H. Samtleben, "Compact and noncompact gauged maximal supergravities in three dimensions," JHEP **0104** (2001) 022 [arXiv:hep-th/0103032].

[78] B. de Wit, H. Nicolai and H. Samtleben, "Gauged Supergravities, Tensor Hierarchies, and M-Theory," JHEP **0802** (2008) 044 [arXiv:0801.1294 [hep-th]].

[79] H. Nicolai and H. Samtleben, "N=8 matter coupled AdS(3) supergravities," Phys. Lett. B **514** (2001) 165 [hep-th/0106153].

[80] B. de Wit, I. Herger and H. Samtleben, "Gauged locally supersymmetric D = 3 nonlinear sigma models," Nucl. Phys. B **671** (2003) 175 [hep-th/0307006].

[81] E. A. Bergshoeff, O. Hohm and T. A. Nutma, "A Note on E(11) and Three-dimensional Gauged Supergravity," JHEP **0805** (2008) 081 [arXiv:0803.2989].

[82] P. S. Howe, J. M. Izquierdo, G. Papadopoulos and P. K. Townsend, "New supergravities with central charges and Killing spinors in (2+1)-dimensions," Nucl. Phys. B **467** (1996) 183 [arXiv:hep-th/9505032].

[83] P. S. Howe and E. Sezgin, "The Supermembrane revisited," Class. Quant. Grav. **22** (2005) 2167 [arXiv:hep-th/0412245].

[84] S. M. Kuzenko, U. Lindstrom and G. Tartaglino-Mazzucchelli, "Off-shell supergravity-matter couplings in three dimensions," JHEP **1103** (2011) 120 [arXiv:1101.4013 [hep-th]].

[85] M. Cederwall, U. Gran and B. E. W. Nilsson, "D=3, N=8 conformal supergravity and the Dragon window," arXiv:1103.4530 [hep-th].

[86] L. Brink and P. S. Howe, "The N=8 Supergravity In Superspace," Phys. Lett. B **88** (1979) 268.

[87] B. de Wit and H. Nicolai, "N=8 Supergravity," Nucl. Phys. B **208** (1982) 323.

[88] P. S. Howe and H. Nicolai, "Gauging N=8 Supergravity In Superspace," Phys. Lett. B **109** (1982) 269.

[89] P. S. Howe, "Supergravity In Superspace," Nucl. Phys. B **199** (1982) 309.

[90] B. de Wit and H. Nicolai, "The Parallelizing S(7) Torsion In Gauged N=8 Supergravity," Nucl. Phys. B **231** (1984) 506.

[91] L. Bonora, P. Pasti and M. Tonin, "Superspace Formulation Of 10-D Sugra+Sym Theory A La Green-Schwarz," Phys. Lett. B **188** (1987) 335.

[92] P. S. Howe, "Pure Spinors Lines In Superspace And Ten-Dimensional Supersymmetric Theories," Phys. Lett. B **258** (1991) 141 [Addendum-ibid. B **259** (1991) 511].

[93] P. S. Howe, "Pure Spinors, Function Superspaces And Supergravity Theories In Ten-Dimensions And Eleven-Dimensions," Phys. Lett. B **273** (1991) 90.

[94] N. Berkovits, "ICTP lectures on covariant quantization of the superstring," arXiv:hep-th/0209059.

[95] N. Berkovits and P. S. Howe, "The cohomology of superspace, pure spinors and invariant integrals," arXiv:0803.3024 [hep-th].

[96] A. A. Rosly, "Super Yang–Mills constraints as integrability conditions," in *Proceedings of the International Seminar on Group Theoretical Methods in Physics*," (Zvenigorod, USSR, 1982), M. A. Markov (Ed.), Nauka, Moscow, 1983, Vol. 1, p. 263 (in Russian); English translation: in *Group Theoretical Methods in Physics*," M. A. Markov, V. I. Man'ko and A. E. Shabad (Eds.), Harwood Academic Publishers, London, Vol. 3, 1987, p. 587.

[97] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, "Unconstrained $\mathcal{N} = 2$ matter, Yang–Mills and supergravity theories in harmonic superspace," Class. Quant. Grav. **1**, 469 (1984).

[98] A. Karlhede, U. Lindstrom and M. Rocek, "Selfinteracting tensor multiplets In $\mathcal{N} = 2$ superspace," Phys. Lett. B **147**, 297 (1984).

[99] B. M. Zupnik and D. V. Khetselius, "Three-dimensional extended supersymmetry in harmonic superspace," Sov. J. Nucl. Phys. **47** (1988) 730 [Yad. Fiz. **47** (1988) 1147].

[100] P. S. Howe and M. I. Leeming, "Harmonic superspaces in low dimensions," Class. Quant. Grav. **11** (1994) 2843 [arXiv:hep-th/9408062].

[101] G. G. Hartwell and P. S. Howe, "(N, P, Q) Harmonic Superspace," Int. J. Mod. Phys. A **10** (1995) 3901 [arXiv:hep-th/9412147].

[102] G. Bossard, P. S. Howe and K. S. Stelle, "On duality symmetries of supergravity invariants," arXiv:1009.0743 [hep-th].

[103] P. Henry-Labordere, B. Julia and L. Paulot, "Real Borcherds superalgebras and M-theory," JHEP **0304** (2003) 060 [arXiv:hep-th/0212346].

[104] E. A. Bergshoeff, M. de Roo, S. F. Kerstan, T. Ortin and F. Riccioni, "IIA ten-forms and the gauge algebras of maximal supergravity theories," JHEP **0607** (2006) 018 [arXiv:hep-th/0602280].

[105] E. A. Bergshoeff, M. de Roo, S. F. Kerstan and F. Riccioni, "IIB Supergravity Revisited," JHEP **0508** (2005) 098 [arXiv:hep-th/0506013].

[106] M. Cederwall, B. E. W. Nilsson and D. Tsimpis, "The structure of maximally supersymmetric Yang-Mills theory: constraining higher-order corrections," JHEP **0106** (2001) 034 [arXiv:hep-th/0102009].

[107] M. Cederwall, B. E. W. Nilsson and D. Tsimpis, "Spinorial cohomology and maximally supersymmetric theories," JHEP **0202** (2002) 009 `hep-th/0110069`.

[108] R. Borcherds, "Generalized Kac-Moody algebras", J.Alg. Vol 115, No. 2, June 1988, p. 501-512.

[109] R. Slansky, "An Algebraic Role For Energy And Number Operators For Multiparticle States," Nucl. Phys. B **389** (1993) 349.

[110] J. Greitz and P. S. Howe, "Maximal supergravity in three dimensions: supergeometry and differential forms," arXiv:1103.2730 [hep-th].

[111] P. S. Howe and P. C. West, Nucl. Phys. B **238** (1984) 181.

[112] J. H. Schwarz, "Covariant Field Equations Of Chiral N=2 D=10 Supergravity," Nucl. Phys. B **226** (1983) 269.

[113] M. Cederwall, A. von Gussich, B. E. W. Nilsson, P. Sundell and A. Westerberg, "The Dirichlet super-p-branes in ten-dimensional type IIA and IIB supergravity," Nucl. Phys. B **490** (1997) 179 [arXiv:hep-th/9611159].

[114] G. Dall'Agata, K. Lechner and M. Tonin, "D = 10, N = IIB supergravity: Lorentz-invariant actions and duality," JHEP **9807** (1998) 017 [arXiv:hep-th/9806140].

[115] N. Berkovits and P. S. Howe, "Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring," Nucl. Phys. B **635** (2002) 75 [arXiv:hep-th/0112160].

[116] I. C. G. Campbell and P. C. West, "N=2 D=10 Nonchiral Supergravity And Its Spontaneous Compactification," Nucl. Phys. B **243** (1984) 112.

[117] J. L. Carr, S. J. J. Gates and R. N. Oerter, "D = 10, N=2a Supergravity In Superspace," Phys. Lett. B **189** (1987) 68.

[118] E. A. Bergshoeff, J. Gomis, T. A. Nutma and D. Roest, "Kac-Moody Spectrum of (Half-)Maximal Supergravities," JHEP **0802** (2008) 069 [arXiv:0711.2035 [hep-th]].

[119] M. Weidner, "Gauged supergravities in various spacetime dimensions," Fortsch. Phys. **55** (2007) 843 [hep-th/0702084].

[120] J. Greitz and P. S. Howe, "Maximal supergravity in D=10: Forms, Borcherds algebras and superspace cohomology," JHEP **1108** (2011) 146 [arXiv:1103.5053 [hep-th]].

[121] V. Kac, "Infinite-dimensional Lie algebras," CUP, 3rd ed. 1990.

[122] S-J. Kang, J-H. Kwon and Y-T Oh, "Peterson-type dimension formulas for graded Lie superalgebras," Nagoya Math J. **163** (2001), 107-144.

[123] A. Iqbal, A. Neitzke and C. Vafa "A Mysterious Duality," Adv. Theor. Math. Phys. **5** (2002) 769 [arXiv:hep-th/0111068v2].

[124] T. Nutma "SimpLie-a simple program for Lie algebras," http://code.google.com/p/simplie/

[125] E. Bergshoeff, J. Gomis, T. Nutma and D. Roest, "Kac-Moody Spectrum of (Half-)Maximal Supergravities," JHEP **0802** (2008) 069 [arXiv:hep-th/0711:2035].

[126] J. Greitz and P. S. Howe, "Unpublished,"